January 1963

A NEW LOOK AT MULTIPLE SCATTERING

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A NEW LOOK AT MULTIPLE SCATTERING

1. The Standard Problem of Multiple Scattering

We shall define the standard problem of multiple scattering by the following set of assumptions, formulated in the conventional language of astrophysics. For a more complete formulation and comments we refer to the list of definitions in section 2.

- Homogeneous plane-parallel atmosphere or slab.
- Optical thickness b; optical depth τ runs from 0 to b.
- Albedo a independent of τ
- Phase function or scattering diagram $\varphi(\alpha)$ of individual particles or volume elements (possibly to include polarization specifications) is independent of τ
- Incident radiation having an arbitrary intensity distribution with angle impinges on one side of the atmosphere; we call this side top, the other side bottom.
- No interdependence of the processes at different wavelengths.
- No internal emission.
- No reflecting bottom surface.
- Steady state (independence of time) and plane waves (independence of coordinates x, y, along the slab).

It is asked to determine the intensity and state of polarization of light emerging under any angle at top (reflected) and at bottom (transmitted).

The word "standard problem" has been occasionally used in this sense but does not belong to the common jargon of astrophysics. It is employed here for convenience and should not be held to imply that the many variations from this problem are somehow less valid. Some such variations are: time-dependent problems; inhomogeneous or curved atmospheres; radiative transfer problems (in which the events at different λ are interlocked by the occurrence of the same temperature distribution).

This entire paper deals with the standard problem. In fact, most of it deals with an even simpler problem, the standard problem for isotropic scattering, defined by the additional assumption that

$$\varphi(\alpha) = 1 .$$

The solution of this problem is well known and its application should present no problems. Yet, the numerical results required in practical problems are still somewhat hard to find. One reason is that most work on the subject contains such lengthy derivations that the "user" has trouble in finding his way. Another reason is that numerical values for the basic functions $X(\mu)$ and $Y(\mu)$ and their moments are available in a very limited number. Even when these are

known, the further substitutions still pose nasty problems. For example, in transmission when $\mu = \mu_0$, the usual formula gives 0/0, and a differentiation of $X(\mu)$ and $Y(\mu)$ with respect to μ is required to find the answer.

The novelty of the present paper is that the relevant equations are written in a new, extremely condensed, manner. This adds nothing of substance but it was felt that the following advantages might result:

(a) More results to a page and quicker reference.

(b) Suitable starting point for machine-programming.

(c) Possibility of following the physical meaning throughout. This is very important because most of the functions to be defined and tabulated have reciprocal physical interpretations corresponding to a reversal of the time direction.

(d) Possibility for derivations which would otherwise become too lengthy. For instance, the second derivation of the formulae for the adding method (section 9) would involve sextuple integrals if written fully.

(e) An exercise toward the more elaborate programs required when the scattering is anisotropic or when polarization is included.

2. Definitions of Physical Quantities and Some Comments

For readers in doubt about or unacquainted with the astrophysical usage, but willing to accept definitions in physical rather than mathematical terms, the following list of definitions and comments may serve to define the problem more clearly. Reference may also be made to the books of Chandrasekhar (1950), Kourganoff (1952) and others. Deviations from commonly accepted usage, where found desirable, have been pointed out specifically.

Radiation or light. These terms are used interchangeably. They may refer to a small or wide wave length region, provided only that the scattering properties (extinction coefficient, albedo, scattering diagram) do not vary with wave length within this region. The amount of radiation or light may be expressed in energetic or luminous units as desired.

Scattering. The process whereby part of the radiation or light arriving at a scattering particle suffers a change in direction, possibly combined with a change in state of polarization, without a change in wavelength. The scattering coefficient k_{sca} will be defined as total scattering cross-section per unit volume, or average cross-section of a particle times number of particles per unit volume. The dimension is length^{-1}. The quantity more commonly used in astrophysics is the mass scattering coefficient, or

total cross-section per unit mass, which is $1/\rho$ times as large, where ρ = density = mass per unit volume. Similarly for absorption and extinction.

Absorption. The process whereby part of the radiation or light arriving at a scattering particle or volume element is absorbed into this particle or volume element and converted into heat. The absorption coefficient is k_{abs} (length^{-1}). The opposite process, emission, generally exists whenever absorption is present but will not be incorporated into our formulae because of one of two reasons:

1. In typical light scattering problems the emission can be ignored because it occurs mostly at quite different wave lengths in the far infrared.
2. In the "grey" radiative transfer problem, when we consider total energy only, the absorption and subsequent emission can be formally combined into one scattering event with albedo 1.

Extinction. Sum of scattering and absorption. Extinction coefficient = $k_{ext} = k_{abs} + k_{sca}$ = relative loss of intensity per unit path length from a rectilinear beam of light.

Albedo. Symbol a. The ratio of scattering coefficient to extinction coefficient; we have

$$k_{sca} = a\, k_{ext}$$

$$k_{abs} = (1 - a)\, k_{ext}$$

The conventional symbol ϖ (curled pi, not omega) for albedo is avoided because most users find it too exotic.

Optical path length. Integral between any two points A and B of geometric path length times local extinction coefficient. No separate symbol is required because the expression $|(\tau_A - \tau_B)/\mu|$, where μ = cosine of angle with normal, suffices.

Optical depth. Optical path length measured perpendicularly to the slab or atmosphere from a convenient zero point (usually taken to be the top surface).

Homogeneous atmosphere. A plane-parallel atmosphere in which a and $\varphi(\alpha)$ do not change with τ. A narrower definition would be to require that each of the coefficients k_{ext}, k_{sca}, and k_{abs} are independent of the geometrical height in the atmosphere, but these coefficients do not appear anywhere in the theory except in the final conversion of optical depth into geometrical depth. Hence the wider definition, which applies pretty well to many practical problems concerning planetary atmospheres, is preferred.

Scattering angle. Symbol α. The angle between the direction of propagation of the light arriving at a particle and the direction

of propagation of the light scattered by it.

Scattering function or scattering diagram. Symbol $\varphi(\alpha)$. The function indicating how the radiation scattered by a single particle or small volume element is distributed with the scattering angle. We adopt the normalization

$$\frac{1}{2}\int_{-1}^{1}\varphi(\alpha)\,d(\cos\alpha) = 1$$

and define the first moment as the asymmetry factor

$$g = \frac{1}{2}\int_{-1}^{1}\varphi(\alpha)\cos\alpha\;\;d(\cos\alpha)$$

In older books $\varphi(\alpha)$ would often be called the phase function (term derived from"phases of the moon", "phases of the planets") and the argument would often be $\pi-\alpha$. In modern work the term forward scattering unambiguously refers to $\alpha = 0$ (or close to 0) but in older work the words forward and backward were sometimes used ambiguously.

In order to describe scattering of polarized light fully, it is necessary to consider each intensity as a four-vector of the four Stokes parameters and the scattering function $\varphi(\alpha)$ as a 4 x 4 matrix. For explicit formulae we refer to Chandrasekhar's book.

Multiple scattering. The successive occurrence of scattering processes by different particles under the assumption that radiation or light arriving at a particle after having been scattered by another one is affected in precisely the same manner as radiation or light coming from a very distant source would be affected. This means that typical vicinity effects are excluded, which is permissible in virtually all applications.

Radiative transfer. The successive occurrence of processes by which not only the direction of the radiation is changed but in which also radiative energy is being converted from one wave length to another wave length. This is usually described as a process of absorption and subsequent re-emission dependent on a locally defined temperature which is determined by the transfer process. We do not consider radiative transfer problems in this paper, except for the equations governing the total energy (integrated over all wave lengths) in the"gray case" absorption coefficient independent of wave length, which happen to be identical for those governing the intensity in multiple scattering with albedo 1 and isotropic scattering diagram .

Intensity. Amount of radiation flowing per unit time per steradian per unit area perpendicular to the direction of propagation. This is the "specific intensity" commonly used in astro-

physics. In the absence of extinction the intensity of radiation emitted by an extended source equals the brightness of that source. It is independent of distance because the area of a beam increases in the same proportion as the solid angle diminishes with increasing distance. The words intensity and brightness can be translated into appropriate standard terms when the system of units has been chosen, but the general definitions suffice for the present paper.

Lambert surface. Surface (of a solid body or other object) which has equal brightness in all directions in the hemisphere from which it can be seen.

Flux or net flux. Amount of radiation flowing per unit time through a unit area parallel to the top or bottom surface of the atmosphere. This definition corresponds with standard usage, but it should be noted that also by standard usage the flux is represented by the product πF, so that $F = \pi^{-1} \times$ flux. The reason for this convention is that $F = 1$, flux $= \pi$, for a Lambert surface with intensity 1. In colloquial astrophysica F is often called the flux; this is very confusing and we shall try to avoid this colloquialism. If μ is the (positive) cosine of the angle between the direction of propagation and the normal, then

$$F(\tau) = 2\int_{-1}^{1} \mu\, I(\tau,\mu)\, d\mu$$

Average intensity. Symbol $\bar{I}$. Intensity averaged over the total solid angle 2π of a hemisphere without weighting factor.

$$\bar{I} = \int_0^1 I(\mu)\, d\mu$$

This concept is useful only for the emergent radiation at top or bottom of the slab. Like F it is an average, or moment, of the emerging intensity distribution, but it gives relatively more weight to the grazing directions than F.

Source function. Symbol J. Amount of radiation or light scattered per unit solid angle from a volume element containing particles with unit total extinction cross-section. This definition is equivalent to saying that the intensity of scattered radiation emerging from a box with surface area $d\bar{\sigma}$ and optical thickness $d\bar{\tau}$ normal to $d\bar{\sigma}$ is $J\, d\bar{\sigma}\, d\bar{\tau}$ (Figure 1).

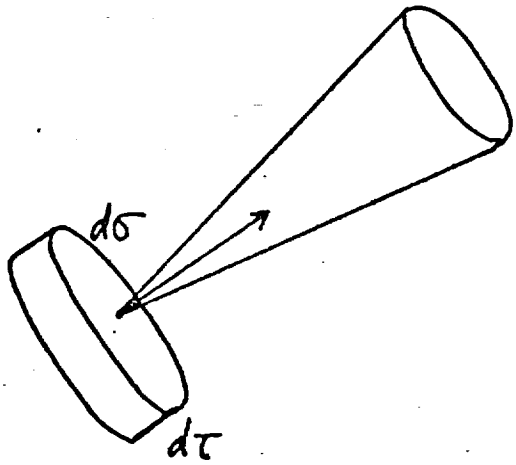


FIGURE 1

J has the dimension of intensity and is independent of direction if the particle scattering is isotropic. However, the definition can be applied also to anisotropic scattering, in which case J depends not only on the position in the medium but also on direction. If polarization is taken into account, J is a 4-vector. For isotropic scattering

$$J(\tau) = \frac{a}{2} \int_{-1}^{1} I(\tau,\mu)\, d\mu$$

Radiation density. Amount of radiative energy per unit volume. If energy units are used to express the intensity, the radiation density involves the same integral as the source function (for isotropic scattering) but the factor is different. Let c be the velocity of light then,

$$\text{Radiation density} = \frac{4\pi}{c\,a} J(\tau)$$

The units could be:

I in erg sec^{-1} cm^{-2} $sterad^{-1}$, J and radiation density in erg cm^{-3}, $c = 3.00 \times 10^{10}$ cm sec^{-1}

or

I in watt m^{-2} $sterad^{-1}$, J and radiation density in joule m^{-3}, $c = 3.00 \times 10^{8}$ m sec^{-1}.

Please note that the integration of μ over two hemispheres, which occurs in the definitions of $F(\tau)$, $J(\tau)$, and the radiation density cannot be expressed in the condensed matrix notation which we shall employ in subsequent sections of this paper.

Reflection function. This function $R(\mu_0, \mu)$ is more completely called the diffuse reflection function. It is defined as the intensity reflected in direction μ when radiation with flux π ($F = 1$) is incident from direction μ_0. Here μ_0 and μ are the cosines of the angles of the direction of incidence and reflection with the normal, taken with positive sign.

Some comments are necessary. First, if the incident radiation comes from a small solid angle, say the Sun's disk with $\Delta\Omega = 0.86 \times 10^{-4}$ steradian, then the average specific intensity incident from directions within this solid angle must be taken as $I_{in} = \frac{\pi}{\mu_0 \Delta\Omega}$ in order to obtain incident flux π ($F_{in} = 1$, see definition above). Hence, the reflected intensity caused by arbitrary incident intensity confined within that solid angle is

$$I_{out} = \frac{\mu_0 \Delta\Omega}{\pi} I_{in} R(\mu_0, \mu) .$$

Secondly, this reflection function is related to Chandrasekhar's function $S(\mu_0, \mu)$ by

$$R(\mu_0,\mu) = \frac{S(\mu_0,\mu)}{4\mu_0\mu}$$

Both functions are symmetric. The reflection formula for arbitrary incident radiation

$$I_{out}(\mu) = \frac{1}{2\mu}\int_0^1 S(\mu_0,\mu)\, I_{in}(\mu_0)\, d\mu_0$$

employed by Chandrasekhar (R. T. page 21, Eq. 122 with minor changes in notation) can be made fully symmetric in two equivalent ways

$$\text{(A)}\quad \{\mu\, I_{out}(\mu)\} = \frac{1}{2}\int_0^1 S(\mu_0,\mu)\, \{\mu_0\, I_{in}(\mu_0)\}\, \frac{d\mu_0}{\mu_0}$$

$$\text{(B)}\quad I_{out}(\mu) = 2\int_0^1 R(\mu_0,\mu)\, I_{in}(\mu_0)\, \mu_0\, d\mu_0$$

We have selected form (B) because the intensities themselves appear and because $R(\mu_0,\mu)$ assumes finite non-zero values for $\mu = 0$ or $\mu_0 = 0$; (but it becomes ∞ for $\mu_0 = \mu = 0$).

Transmission function. The function $T(\mu_0,\mu)$ is more completely called the diffuse and direct transmission function. It is defined as the intensity emerging in direction μ from the

bottom of a plane-parallel atmosphere when radiation with flux π ($F = 1$) is incident on the top from direction μ_0. Here μ_0 and μ are again positive cosines of the angles with the normal. Included in this definition is the direct radiation, i.e., the part of the incident radiation which emerges at the bottom without scattering. This is convenient, first because the formulae are simpler that way and, secondly, because in many experimental conditions the direct and scattered radiation are indistinguishable.

3. The Milne Equation

Sections 3 to 5 refer only to the case of isotropic scattering [$\varphi(\alpha) = 1$]. We repeat the main notations.

b = total optical depth of layer
a = albedo of particles
} regarded constant

τ or τ' = optical depth from top surface

μ_0 = positive cosine of angle of incidence measured from normal

μ = positive cosine of angle of emergence at top or bottom surface, measured from normal

"top" = side from which incident radiation comes

"bottom" = the other side

The preceding definitions make clear that the source function $J(\tau)$ and the intensity $I(\tau,\mu)$ are expressible in terms of each other, namely

J in terms of I by an integration over directions (μ)

I in terms of J by an integration over depth (τ)

From here we can choose to eliminate J, obtaining one equation, the equation of transfer, for I. Alternatively we can choose to eliminate I, obtaining one equation, the Milne equation, for J.

We choose the second alternative, which is distinctly simpler because (only for isotropic scattering!) a function of one variable instead of two is involved. This means that inside the atmosphere we consider functions of τ only and at top and bottom we consider functions of μ only. Incidentally this relieves us from the always somewhat clumsy notations to distinguish between intensity up or down. As it is, we can take μ always positive corresponding to the cosine of the angle with the outward normal, the term "outward" being unambiguous at top or bottom but not at any intermediate point.

Let the incident radiation have the intensity $I_{in}(\mu_o)$, independent of the azimuth angle φ. If this assumption is not made, one further (trivial) integration over φ is involved.

The emergent intensities $I_{out}(\mu)$ must then be found from the following set of equations:

$$(1) \qquad J(\tau) = \frac{a}{2}\int_0^1 I_{in}(\mu_0)\, e^{-\tau/\mu_0}\, d\mu_0 + \frac{a}{2}\int_0^b E_1(|\tau-\tau'|)\, J(\tau')\, d\tau'$$

$$(2) \qquad \begin{cases} I_{out,\,refl}(\mu) = \displaystyle\int_0^\tau J(\tau')\, e^{-\tau'/\mu}\, \frac{1}{\mu}\, d\tau' \\ I_{out,\,transm}(\mu) = I_{in}(\mu)\, e^{-b/\mu} + \displaystyle\int_0^\tau J(\tau')\, e^{-(b-\tau')/\mu}\, \frac{1}{\mu}\, d\tau' \end{cases}$$

These are familiar equations in an unfamiliar setting, because of the assumption of a diffuse incident radiation field. The main problem, of course, is solving for $J(\tau)$ from the first integral equation, which is the Milne equation. The kernel contains the exponential integral

$$E_1(y) = \int_1^\infty e^{-yu}\, \frac{du}{u}$$

The flux can be similarly expressed. Writing the downward net flux as $\pi F(\tau)$ we obtain

$$(3) \qquad F(\tau) = 2\int_0^1 I_{in}(\mu_0)\, e^{-\tau/\mu_0}\, \mu_0\, d\mu_0 + 2\int_0^\tau E_2(\tau-\tau')\, J(\tau')\, d\tau' - 2\int_\tau^b E_2(\tau'-\tau)\, J(\tau')\, d\tau',$$

where

$$E_2(y) = \int_1^\infty e^{-yu} \frac{du}{u^2}$$

As a check we differentiate equation (3) and find

$$\frac{dF(\tau)}{d\tau} = 4\left(1 - \frac{1}{a}\right) J(\tau) \tag{4}$$

It is easily verified from the definitions that upon multiplication by $\pi\, d\tau$ we have left and right the amount of radiation absorbed in a volume element with area 1 and thickness $d\tau$ at depth τ.

4. Matrix Notation

The preceding equations and the various solution methods which follow contain integrations of only two types:

integration over τ or τ' from 0 to b

integration over μ or μ_o from 0 to 1

We now introduce a condensed notation in which these integrations appear as matrix products. A function of one variable will be called a μ-vector or a τ-vector. A function of two variables will be called a $\mu\mu$-matrix, a $\mu\tau$-matrix, a $\tau\mu$-matrix or a $\tau\tau$-matrix. The order has to be strictly observed. Multiplication of two matrices or a matrix and a vector, or two vectors is a short notation for

integration over the adjoining arguments, which should be both τ or both μ . We adopt the definitions:

τ -multiplication $$FG = \int_0^b F(\tau)\, G(\tau)\, d\tau$$

μ -multiplication $$F'G' = \int_0^1 F'(\mu)\, G'(\mu)\, 2\mu\, d\mu$$

where $F(\tau)$, $G(\tau)$, $F'(\mu)$, $G'(\mu)$ are arbitrary functions.

These multiplications obey all rules of matrix multiplication, in particular the associative property. The factor 2μ in the definition of the μ -multiplication is necessary in order to make the matrices symmetric (compare the remarks made on $R(\mu_0,\mu)$ in section 2). Not a single transformation or approximation is made at this stage. We go on working with continuous variables and have changed only the notation. Evidently, actual matrices with a finite set of numbers may be employed in machine computations, but we are not yet concerned with the technique for doing so or with the accuracy lost in that process.

The curious fact about this notation is that it seems <u>too</u> economic, for the notation does not tell whether a function of μ or μ_0 is meant or whether the argument is τ or τ'. This will always be clear from the context, i.e., from the physical meaning

of the expression. Nor do the $\mu\mu$ and $\tau\tau$-matrices specify in what order the arguments are to taken, for instance, in the order (μ, μ_0) or (μ_0, μ). This will always be irrelevant because of symmetry. These curious properties are manifestations of the reciprocity principle, which generally expresses the possibility of inverting the time order of events. Several consequences are mentioned explicitly in further sections.

A formal distinction between the two matrices representing the function

$$\frac{e^{-\tau/\mu}}{\mu} = \begin{cases} P & (\mu\tau\text{-matrix}) \\ \overline{P} & (\tau\mu\text{-matrix}) \end{cases}$$

is necessary in this notation but ceases to be visible when the integrations are written out. $\overline{P}$ is the transposed matrix of P. Similarly, $\overline{Q}$ is the transposed matrix of Q. All square matrices ($\tau\tau$ or $\mu\mu$-matrices) which occur in our formulae are symmetric, i.e., their own transpose.

Here follows the list of the vectors and matrices which we shall use.

	Symbol	Function
μ-vectors	I	$I_{in}(\mu_0)$ = incident intensity
	O	$I_{out}(\mu)$ = emergent intensity

	Symbol	Function
μ -vectors (contd)	D	$(2\mu)^{-1}$ x angular detector response
	U	1
	N	$\frac{1}{2\mu}$
	Z	operator defined by $Z\Phi = \Phi(0)$, Φ arbitrary
τ -vectors	J	$J(\tau)$ = total source function
	S	$aJ_1(\tau)$ = first order source function
	J_p	$J_p(\tau)$ = p-th order source function for a = 1 (p = 1, 2, 3, ...)
	W	1
	A	operator defined by $A\Phi = \Phi(0)$, Φ arbitrary
	B	operator defined by $B\Phi = \Phi(b)$, Φ arbitrary
	E	$E_1(\tau)$
	G	$2E_2(\tau)$
$\mu\mu$ -matrices	1	$\frac{1}{2\mu}\delta(\mu,\mu_o)$, diagonal matrix, operator converting only the argument μ_o into μ, or conversely.

	Symbol	Function
$\mu\mu$ -matrices (contd)	T_o	$\frac{1}{2\mu} e^{-b/\mu} \delta(\mu,\mu_o)$, diagonal matrix, operator converting a function of μ_o into the same function of μ multiplied by $e^{-b/\mu}$
	R	reflection function $R(\mu_o,\mu)$
	T	transmission function $T(\mu_o,\mu)$
$\tau\tau$ -matrices	1	$\delta(\tau,\tau')$, diagonal matrix, operator converting only the argument τ' into τ, or conversely
	M	$\frac{1}{2}E_1(\lvert\tau-\tau'\rvert)$ = Milne operator. This operator is commonly known as the Λ-operator.
	C	$(1 - aM)^{-1}$ = the complete redistribution function

$\mu\tau$ -matrices, Symbol	Function	$\tau\mu$ -matrices, Symbol
P	$\frac{1}{\mu} e^{-\tau/\mu}$	$\overline{P}$
Q	$\frac{1}{\mu} e^{-(b-\tau)/\mu}$	$\overline{Q}$

Several singular functions occur in this tabulation. They are all common delta-functions and obviously indispensable in this type of calculus. Their meaning may be described in two ways.

(a) As an operator, i.e., by stating that matrix multiplication with this function is an order to replace one variable by another variable, or by a specific number, as specified.

(b) As a continuous function which has large non-zero values only in a very narrow range of values of the independent variable and which, upon integration by the general definition of matrix multiplication, yields the specified result in the limit in which this interval shrinks to 0.

The definition (b) is a little more cumbersome but corresponds to actual situations, e.g., illumination by the Sun, which is not a point source, or observation by means of a telescope with limited resolving power directed at the limb of a planet's disk. In these situations the angular integration interval is small but not strictly zero.

Several relations between these singular matrices may be noted. They are trivial, mathematically, but a loose description of their physical meaning may have some interest.

First,

$$\bar{P}Z = ZP = A$$

$$\bar{Q}Z = ZQ = B$$

Verification proceeds by multiplying with an arbitrary τ-vector S and writing the integrals out, for instance, ZPS = Z(PS) = S(0) = AS. The descriptive value of these relations is: "light incident under a grazing angle penetrates (without scattering) only into the topmost layers of the atmosphere", or reciprocally, "viewing an atmosphere under grazing angles we observe only the topmost layers." The interpretation of the second relation is quite similar.

Secondly,

$$2AM = 2MA = NP = \bar{P}N = E_1(\tau)$$

$$2BM = 2MB = NQ = \bar{Q}N = E_1(b - \tau)$$

The verification is straightforward. The descriptive meaning of the top relation is that radiation from a narrow layer of isotropic sources on top of the atmosphere (N) is first scattered ($\bar{P}$) in different layers distributed as the Milne distribution function (M), measured from the top down (A). Reciprocally, a radiation detector with isotropic sensitivity characteristic (N) held at the top of the atmosphere will record the radiation from various

layers (P) weighted according to the Milne distribution function (M), measured from the top down (A). The interpretation of the bottom relation is entirely similar.

Thirdly, some related products are

$$2WMA = 2WMB = 2AMW = 2BMW = 1 - E_2\ (b)$$

$$2AMB = 2BMA = E_1(b)$$

$$2UM = 2MU = 2 - E_2\ (\tau) - E_2\ (b - \tau)$$

2AMA and 2BMB diverge

5. Internal Relations

It is possible to avoid in an elegant manner any detailed discussion of what happens to the radiation inside the layer and to concentrate on the equations governing the external radiation field (external relations, see 6). Although this method has been brought to prominence by Chandrasekhar, we feel that logical priority should be given to a method in which we follow "what happens to the radiation" when it gets inside and how it gets out again. This we shall do in the present section. Of course, none of this is new. Only the presentation may be more concise and, we hope, more transparent than in most textbooks.

The following derivation will be confined to isotropic

scattering, $\varphi(\alpha) = 1$.

The events in a logical order, are:

First Event: The radiation incident from the top, I, which has an arbitrarily given intensity distribution with angle, penetrates into the layer and is scattered at various depths, thus establishing the original (first-order) source function

$$S = \frac{a}{4} \bar{\bar{P}} I$$

The factor $\frac{1}{4}$ appears when the integral is written down fully employing the definitions given in Section 2. Essentially it is due to the fact that the flux has a factor π but that scattering occurs into 4π steradians.

Second Event: The radiation forthcoming from the source distribution S is scattered again and again, thus establishing a complete source function, which includes S and all higher orders and is written as

$$J = CS$$

Third Event: Radiation from this source function at various depths reaches the top surface under various angles and gets out, giving the emerging intensity

$$O_{top} = PJ$$

Similarly, the radiation from these sources emerging at the bottom is QJ but in addition, some of the original incident radiation shines through without any scattering, giving at the bottom the intensity $T_o I$. The combined emerging radiation intensity is

$$O_{bottom} = T_o I + QJ$$

The same equations, written fully with integrals, were given as Equations 2 in Section 3.

Fourth Event: If a detector is used to measure the emerging radiation (at top or bottom) sampling the different angles with a certain weighting factor described by D, the detector reading is a number

$$d = DO$$

The combined result of these successive matrix multiplications thus assumes the form of a reflection function R and a transmission function T operating on I.

$$O_{top} = RI = PC\tfrac{1}{4}a\bar{P}I$$

$$O_{bottom} = TI = T_o I + QC\tfrac{1}{4}a\bar{P}I$$

So that we have the equations

$$R = PC\tfrac{1}{4}a\bar{P}$$

$$T = T_o + QC\tfrac{1}{4}a\bar{P} \ ,$$

which may be taken as the definitions of R and T in mathematical terms. Granting the symmetry of C (see below) it is readily seen that both R and T are symmetric matrices.

The corresponding detector readings can be written as

$$d_{top} = DRI = DP\ C\tfrac{1}{4}a\bar{P}I$$

$$d_{bottom} = DTI = DT_oI + DQ\ C\tfrac{1}{4}a\bar{P}I$$

In these equations D occupies a position symmetric to I and DP (or DQ) a position symmetric to PI. These formulae may, therefore, be turned around and may be given a different (reciprocal) interpretation in which I refers to the detector characteristic, D to the incident intensity distribution. Particular choices for D and I and their reciprocal interpretations will be discussed in the next section. The advantages of the matrix notation are that these symmetries appear quite naturally and that the choice of the order of computation may obviously be decided on the basis of convenience only. For instance, d_{top} may be computed from

$$d_{top} = D(PC\tfrac{1}{4}a\bar{P})I = DRI$$

or from

$$d_{top} = (DP)\ C(\tfrac{1}{4}a\bar{P}I)$$

whichever seems more convenient in a particular application.

The determination of C forms the core of the multiple scattering problem. This subject is important enough to describe two separate ways of deriving the equations.

First, if we refuse to recognize any hierarchy of first, second and higher-order scattering, we must reason as follows. The local intensity at any depth τ and hence the local source function J comes partly from incident radiation which has penetrated to that depth (this part we have called S) and partly from radiation scattered by the atmosphere, which itself depends on J at all other depths τ'. This is expressed by the matrix equation (fully written as Equation 1 in Section 3)

$$J = S + aMJ$$

Consequently,

$$S = (1 - aM)J$$

and

$$J = CS\ , \qquad C = (1 - aM)^{-1}$$

In the second derivation we consider that radiation from outside gives S, the first-order part of the source function.

One subsequent scattering gives second-order sources distributed as aMS. The third order requires again multiplication by aM. Hence the total sum is

$$J = S + aMS + a^2M^2S + \ldots = CS$$

where

$$C = 1 + aM + a^2M^2 + \ldots = (1 - aM)^{-1}$$

Both derivations are fully equivalent. The second one suggests at once a method for numerical computation of C, more details about which are given in Section 8. This is by no means the only one possible.

It may be noted that zero-order sources, i.e., isotropically emitting sources of radiation embedded within the atmosphere with an arbitrary distribution $S_o(\tau)$ can be easily incorporated. The only change is that S should be replaced by $S + S_o$.

6. External Relations

External relations are those in which no functions of τ occur. Several authors, most completely Chandrasekhar, have shown that these external relations can be derived by means of invariance principles without ever going through the calculation of the internal intensity or source function. This can be done

for an arbitrary scattering diagram of the single particles. The external relations then involve μ_0, μ and the azimuth angles φ_0 and φ. The relations are simplest for isotropic scattering, in which case only μ_0 and μ are involved and the results can be expressed in terms of two functionc $X(\mu)$ and $Y(\mu)$ and their moments.

However, before we come to these results, we wish to summarize the more general relations which can be written down without specifying the precise form of the deflection and transmission function of R and T. Any measurement of the emerging radiation at the illuminated side of the slab (top), or at the other side (bottom), has the form of a "detector reading"

$$d = \text{DRI (top)}$$

or

$$d = \text{DTI (bottom)}$$

These expressions are the product of a μ-vector, a $\mu\mu$-matrix and a μ-vector and thus form external relations. If we avoid specifying the form of R, these relations have a far more general applicability than those in which R and T are specified by the formulae derived in the preceding section. For instance, they may refer to an inhomogeneous slab in which the albedo depends on depth. They hold similarly if the scattering diagram is anisotropic, perhaps even varying with depth, and the incident

radiation (and hence also the emergent radiation) is independent of azimuth. Obvious extensions to include the azimuth dependence might be made but we shall not do this. Hence we work with the reflection and transmission functions $R = R(\mu, \mu_0)$ and $T = T(\mu, \mu_0)$ and for the moment it suffices that we postulate their symmetry on the basis of the reciprocity principle.

We may choose for I either an operator stating that we have to take a particular value of μ_0, which corresponds to a plane wave incident under one angle. Or, we may choose for I a function representing the incidence of light from a range of different directions. Similarly, we may choose to specify D in such a manner that we measure radiation emerging under a specific angle (the cosine of which is μ) or we may take it to be a non-singular function so that we measure an integral of the emergent light with a well-defined weighting function for the angles. We have avoided the introduction of separate symbols for the vector-operators specifying a particular angle (μ_0 or μ), for they do not involve a separate step in the computation, once the function to which they are applied has been tabulated. Only the specification that μ_0 or μ is 0 (grazing angles) has been introduced as the vector Z.

Functions I and D specifying integrals can, of course, be

chosen in infinite variety. For mathematical and physical reasons we choose two simple ones, N and U to be applied as a matter of routine in all tabulations alongside the values of the functions R and T for chosen values of μ_0 and μ. This is thought to be an improvement in convenience with respect to older papers where these integrals, if given at all, often are at quite different places from the function values.

The following integrals over the emergent radiation O are obtained:

$$NO = \int_0^1 O(\mu)\,d\mu = \text{average intensity}$$

$$UO = \int_0^1 O(\mu)\,2\mu\,d\mu = \text{emerging flux } / \pi$$

Multiplication by N and U thus defines the zero and first moment, normalized in such a manner that an equal-intensity distribution O = 1 gives NO = 1 and UO = 1. They can also be regarded as differently wieghted averages of O in the domain, μ = 0 to 1, on which O is defined. The physical meaning of NO is $c/2\pi$ times the radiation density of the emerging radiation, as defined in Section 2.

In the reciprocal formulae N or U specify the distribution of incident radiation I and the physical interpretation is different.

$I = N = 1/(2\mu)$ means that the radiation intensity increases as $1/\mu$ towards the grazing angles. This is the intensity distribution that would be forthcoming from a narrow layer of isotropically emitting sources. The incident flux is π . The formula $I = U = 1$ means that the radiation is independent of direction. This occurs, e.g., in the emission by a black body, or in the light diffusely reflected from an ideal white surface; the classical term for this distribution is Lambert's law. The incident flux is again π .

In Tables 1 to 4 we summarize the various combinations to which these choices for D and I lead. The values or functional forms for the simple products are written in the tables. Those for the products involving R and T will be specified presently. The values for the directly transmitted light change into those for the direct light as $b \rightarrow 0$.

Tables 1 and 2. Radiation Field at Top

	Incident (flux down) Values of DI			Diffusely reflected (flux up) Values of DRI		
	incident intensity any direction	incident average intensity	incident flux $/\pi$	reflected intensity any direction	reflected average intensity	reflected flux $/\pi$
incidence from one direction	identity	$N = \frac{1}{2\mu_0}$	$U = 1$	R	NR	UR
incidence from narrow source layer	$N = \frac{1}{2\mu}$	$NN = \infty$	$UN = 1$	RN	NRN	URN
incidence from Lambert surface	$U = 1$	$NU = 1$	$UU = 1$	RU	NRU	URU

Tables 3 and 4. Radiation Field at Bottom

	Directly transmitted Values of DT_oI			Transmitted, including direct part Values of DTI		
	directly transmitted intensity any direction	directly transmitted average intensity	directly transmitted flux	transmitted intensity any direction	transmitted average intensity	transmitted flux
incidence from one direction	T_o	$NT_o = \frac{e^{-b/\mu_o}}{2\mu_o}$	$UT_o = e^{-b/\mu_o}$	T	NT	UT
incidence from narrow source layer	$T_oN = \frac{e^{-b/\mu}}{2\mu}$	$NT_oN = \frac{1}{2}E_1(b)$	$UT_oN = E_2(b)$	TN	NTN	UTN
incidence from Lambert surface	$T_oU = e^{-b/\mu}$	$NT_oU = E_2(b)$	$UT_oU = 2E_3(b)$	TU	NTU	UTU

We now return to the consideration of homogeneous atmospheres with isotropic scattering, in which the albedo a and the optical thickness b are the only parameters appearing in the external relations beside the cosines μ_0 and μ. Under this assumption the familiar way of expressing both R and T and their various integrals listed in Tables 1 to 4 is by means of two functions $X(\mu)$ and $Y(\mu)$ and their moments

$$\alpha_p = \int_0^1 X(\mu)\,\mu^p\,d\mu \qquad p = 0, 1, \ldots$$

$$\beta_p = \int_0^1 Y(\mu)\,\mu^p\,d\mu \qquad p = -1, 0, 1, \ldots$$

For a definition of these functions we may use

$$NR = RN = \frac{1}{2\mu}\left\{X(\mu) - 1\right\}$$

$$NT = TN = \frac{1}{2\mu}\,Y(\mu)$$

For detailed derivations we refer to the literature. It may be shown that

$$R = \frac{a}{4(\mu+\mu_0)} \left\{ X(\mu)X(\mu_0) - Y(\mu)Y(\mu_0) \right\}$$

$$T = \frac{a}{4(\mu_0-\mu)} \left\{ X(\mu)Y(\mu_0) - Y(\mu)X(\mu_0) \right\} + T_0$$

$$UR = RU = r(\mu) = 1 - (1 - \tfrac{a}{2}\alpha_0) X(\mu) - \tfrac{a}{2}\beta_0 Y(\mu)$$

$$UT = TU = t(\mu) = \tfrac{a}{2}\beta_0 X(\mu) + (1 - \tfrac{a}{2}\alpha_0) Y(\mu)$$

$$URN = NRU = \alpha_0 - 1$$

$$UTN = NTU = \beta_0$$

$$URU = r_C = 1 - 2\alpha_1 + a(\alpha_0\alpha_1 - \beta_0\beta_1)$$

$$UTU = t_C = 2\beta_1 + a(\beta_0\alpha_1 - \alpha_0\beta_1)$$

$$\text{NRN} = \int_0^1 \frac{1}{2\mu} \{ X(\mu) - 1 \} \, d\mu = \tfrac{1}{2} \alpha_{-1}^*$$

This integral is the non-divergent part of $\frac{1}{2}\alpha_{-1}$.

$$\text{NTN} = \tfrac{1}{2} \beta_{-1}$$

Finally, the fractions of the incident flux which are absorbed inside the atmosphere follow by subtracting from 1 the sum of the fluxes emerging at top and bottom. The results are given in Table 5.

Table 5. Absorbed Fraction of Incident Flux

incident from	absorbed fraction of incident flux
one direction	$\text{U} - \text{UR} - \text{UT} = (1 - \frac{a}{2}\alpha_0 - \frac{a}{2}\beta_0) \{ X(\mu_0) - Y(\mu_0) \}$
narrow source layer	$1 - \text{URN} - \text{UTN} = 2 - \alpha_0 - \beta_0$
Lambert surface	$1 - \text{URU} - \text{UTU} = (2 - a\alpha_0 - a\beta_0)(\alpha_1 - \beta_1)$

For comparison, it may be noted that the result for a narrow source layer may also be written in the form

$$(1 - \tfrac{a}{2}\alpha_0 - \tfrac{a}{2}\beta_0)(\alpha_0 - \beta_0),$$

which is equal to the expression given in virtue of the identity

$$\left(\frac{a}{2}\alpha_0 - 1\right)^2 - \left(\frac{a}{2}\beta_0\right)^2 = 1 - a$$

Chandrasekhar has chosen these non-linear equations as the defining equations of $X(\mu)$ and $Y(\mu)$. This choice was somewhat unfortunate, because the existence of a unique solution (for $a \neq 1$) is not obvious, although numerically the equations can be used in an efficient iteration method. A number of papers have been devoted to this uniqueness problem and it can be considered solved now (see Busbridge's book). In the conservative case $a = 1$ the solutions become undetermined.

These problems are avoided if we use consistently the definitions of $X(\mu)$ and $Y(\mu)$ in terms of NR and NT. These definitions correspond to very simple physical definitions as follows (van de Hulst 1947):

> Place a narrow source layer in front of a plane-parallel atmosphere. The radiation seen in direction μ is thereby multiplied by a factor $X(\mu)$.
>
> Place a narrow source layer behind a plane-parallel atmosphere. The radiation seen in direction μ is thereby multiplied by a factor $Y(\mu)$.

Substituting for R and T the expressions from section 5 we obtain the definitions of $X(\mu)$ and $Y(\mu)$ in terms of known matrices:

$$X(\mu) = 1 + \frac{a}{2}\,\mu\, NPC\bar{P} = 1 + \frac{a}{2}\mu PC\bar{P}N$$

$$Y(\mu) = e^{-b/\mu} + \frac{a}{2}\,\mu\, NQC\bar{P} = e^{-b/\mu} + \frac{a}{2}\mu QC\bar{P}N,$$

which leave no worries about existence or uniqueness, once it has been established that the matrix $C = (1 - aM)^{-1}$ exists. The conservative case is in no way exceptional in this formulation (except for $b = \infty$, where $a = 1$ is at the same time the critical value and some caution may be required).

Chandrasekhar happened to choose among the various solutions of the non-linear equations for $X(\mu)$ and $Y(\mu)$ in the conservative case a "standard" set which does not correspond to the linear definition given above, or any of its equivalent forms. The following relations, derived by Chandrasekhar, may be used to obtain the correct functions from the standard set.

functions:
$$\begin{cases} X(\mu) = X_{st}(\mu) + Q\mu\{X_{st}(\mu) + Y_{st}(\mu)\} \\ Y(\mu) = Y_{st}(\mu) - Q\mu\{X_{st}(\mu) + Y_{st}(\mu)\} \end{cases}$$

moments:
$$\begin{cases} \alpha_p = \alpha_p^{st} + Q(\alpha_{p+1}^{st} + \beta_{p+1}^{st}) \\ \beta_p = \beta_p^{st} - Q(\alpha_{p+1}^{st} + \beta_{p+1}^{st}) \end{cases}$$

The function $Q = Q(t)$ was defined and tabulated by Chandrasekhar.

7. Low Order Scattering

The expansion of C in the form

$$C = 1 + aM + a^2M^2 + a^3M^3 + \cdots$$

makes it possible to expand any quantity F in which this linear operator is a factor in a power series in a of the form

$$F(a) = F_o + aF_1 + a^2F_2 + a^3F_3 + \cdots$$

We shall call a^pF_p the term corresponding to p-th order scattering. In this manner terms of different order may be distinguished in intensity, flux, source function, or radiation density. The coefficients of the power series expansion for the source function J in the standard problem are

$$J_o = 0 \quad \text{(assumed absence of embedded light sources)}$$

$$J_1 = \frac{S}{a} = \frac{1}{4}\,\overline{PI}$$

$$J_p = M^{p-1}\,J_1 \qquad (p \geq 1)$$

The total source function accordingly is

$$J = \sum_{p=1}^{\infty} a^p J_p = \sum_{p=1}^{\infty} (aM)^{p-1} S = CS$$

The corresponding emergent radiation of order $p \geq 1$ is specified by Table 6.

Table 6

emerging radiation of order $p \geq 1$		reflected	transmitted
intensity $\mu \neq 0$	O_p	PJ_p	QJ_p
intensity $\mu = 0$	ZO_p	$AJ_p = J_p(0)$	$BJ_p = J_p(b)$
average intensity	NO_p	$2AJ_{p+1} = 2J_{p+1}(0)$	$2BJ_{p+1} = J_{p+1}(b)$
flux/π	UO_p	$GJ_p = \int_0^b 2E_2(\tau) J_p(\tau) d\tau$	$\int_0^b 2E_2(b-\tau) J_p(\tau) d\tau$

It has decided advantage (in connection with numerical checks and with the discussion of optically thin atmospheres) to know the results for several orders separately. Only those for the first order and some for the second order can be expressed in terms of the exponential integrals,

$$E_p(x) = \int_1^\infty \frac{e^{-xt}}{t^n}\, dt \ ,$$

the "second order" exponential integrals

$$E_1^{(2)}(x) = \int_x^\infty E_1(t)\, \frac{dt}{t}$$

and related functions. We summarize the results which have been derived elsewhere (Van de Hulst 1947, Kourganoff 1953).

Zero Order. Does not occur in reflection. For transmission, see Table 3 (page 35).

First order. The following combinations occur (Table 7), with top values (reflection) and bottom values (transmission) written together in each box.

Table 7

I	$\frac{S}{a} = J_1 = \frac{1}{4}PI$	emerging intensity	emerging average intensity	emerging flux/π
Incidence from direction μ_0	$\frac{1}{4}P = \frac{e^{-\tau/\mu_0}}{4\mu_0}$	R_1 T_1	NR_1 NT_1	UR_1 UT_1
Incidence from Narrow source layer	$\frac{1}{4}E = \frac{1}{4}E_1(\tau)$	R_1N T_1N	NR_1N NT_1N	UR_1N UT_1N
Incidence from Lambert surface	$\frac{1}{4}G = \frac{1}{2}E_2(\tau)$	R_1U T_1U	NR_1U NT_1U	UR_1U UT_1U

The complete results are collected in Table 8. The reciprocal expressions (e.g. NR_1 instead of R_1N) can be obtained by simply replacing μ by μ_0 and conversely.

Table 8. Analytic Expressions for First-order Scattering.

	for $\mu \neq 0$	for $\mu = 0$
R_1	$\frac{1}{4(\mu+\mu_0)} \left\{ 1 - e^{-b\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)} \right\}$	$\frac{1}{4\mu_0}$
T_1	$\frac{1}{4(\mu-\mu_0)} \left\{ e^{-b/\mu} - e^{-b/\mu_0} \right\}$	$\frac{1}{4\mu_0} e^{-b/\mu_0}$
diagonal values $\mu=\mu_0$: R_1	$\frac{1}{8\mu} \left\{ 1 - e^{-\frac{2b}{\mu}} \right\}$	∞
T_1	$\frac{b}{4\mu^2} e^{-b/\mu}$	0
R_1N	$\frac{1}{4\mu} F_1\left(-\frac{1}{\mu}, b\right)$	∞
T_1N	$\frac{1}{4\mu} e^{-b/\mu} F_1\left(\frac{1}{\mu}, b\right)$	$\frac{1}{4}E_1(b)$
R_1	$\frac{1}{2\mu} F_2\left(-\frac{1}{\mu}, b\right)$	$\frac{1}{2}$
T_1	$\frac{1}{2\mu} e^{-b/\mu} F_2\left(\frac{1}{\mu}, b\right)$	$\frac{1}{2}E_2(b)$

$$NR_1N = \frac{1}{4} G_{11}(b) \qquad NT_1N = \frac{1}{4} G'_{11}(b)$$

$$NR_1U = \frac{1}{2} G_{12}(b) \qquad NT_1U = \frac{1}{2} G'_{12}(b)$$

$$UR_1U = G_{22}(b) \qquad UT_1U = G'_{22}(b)$$

The equation for R_1 is the famous Lommel-Seeliger formula, often employed in astrophysics about the turn of the century. The other formulae are given in the notation of Van de Hulst (1947), although the F-functions had been studied as early as 1913 by King. The notations of Chandrasekhar and Kourganoff are identical, except for the arguments of the F-functions, which they write (τ, μ) instead of $(\frac{1}{\mu}, \tau)$. The defining equations are

$$F_n(s, b) = \int_0^b e^{s\tau} E_n(\tau)\, d\tau$$

$$G_{nm}(b) = G_{mn}(b) = \int_0^b E_m(\tau)\, E_n(\tau)\, d\tau$$

$$G'_{nm}(b) = G'_{mn}(b) = \int_0^b E_m(\tau)\, E_n(b-\tau)\, d\tau$$

For tables, special values, and further properties we refer to the literature.

Second order. Second order scattering generally involves even more complicated functions. It would not pay to write complete formulae.

Van de Hulst (1947) and Chandrasekhar (1950) have discussed formulae which may serve to calculate some further quantities connected

with the second-order scattering in an approximate fashion. In view of the availability of numerical tables now being constructed there is no need to repeat these approximate equations.

The second-order values of a product containing the factor Z (grazing incidence or grazing reflection) can be obtained from the first-order values of the corresponding product containing the factor N (see next section).

8. Grazing Angles

It is obvious that several of the results which we have derived must be considerably simplified if we specify

$$\text{grazing incidence,} \quad \mu_0 = 0, \quad I = Z$$

or

$$\text{grazing emergence,} \quad \mu = 0, \quad D = Z$$

We shall now see what simplifications result in this manner. The resulting formulae are useful in various ways, and it seemed useful to have them collected in this separate section.

ing incidence, $\mu_0 = 0$, I = Z. The first-order source function

$$S = \tfrac{1}{4}a\bar{P}I = \tfrac{1}{4}a\bar{P}Z = \tfrac{1}{4}aA.$$

narrow source layer on top of the atmosphere sending a flux
n the form of scattered light of first order and a flux $\tfrac{1}{2}\pi a$
the atmosphere. The latter part yields reflection (second
higher) and transmission (first order and higher) by pre-
e same course of events that would have followed illumination
ow source layer above the atmosphere, I = N. Only the factor
be added. Hence we have

$$RZ = \frac{a}{2}N + \frac{a}{2}RN = \frac{a}{4\mu}X(\mu)$$
$$TZ = \frac{a}{2}TN = \frac{a}{4\mu}Y(\mu)$$

ssions in terms of $X(\mu)$ and $Y(\mu)$ follow at once from the
equations of these functions on page 36.

ing emergence, $\mu = 0$, D = Z. The situation is entirely
The reciprocal form of the same equations is

$$ZR = \frac{a}{2}N + \frac{a}{2}NR = \frac{a}{4\mu}X(\mu)$$
$$ZT = \frac{a}{2}NT = \frac{a}{4\mu}Y(\mu)$$

A formal derivation of the top equations may, for instance, be obtained by writing identities derived in the preceding sections in the form

$$NRI = NPJ = 2AMJ = \frac{2}{a}A(J-S)$$

$$\frac{2}{a}AS = \tfrac{1}{2}A\bar{P}I = NI$$

$$ZRI = ZPJ = AJ$$

Upon combining these and omitting the factor I, the top equation follows. Another derivation may be obtained from Table 5, where it was shown that both for absorption and transmission

$$ZO_p = \tfrac{1}{2}NO_{p-1} \qquad (p \geqslant 1)$$

Observe further that

$$\text{in reflection: } ZO_o = o,\ aZO_1 = ZPS = \frac{a}{2}NI$$

$$\text{in transmission: } ZO_o = o,\ aZO_1 = \frac{a}{2}NO_o.$$

Multiply the equality by a^p and add from p=1 to ∞ and, after omission of the arbitrary incidence vector I, again the same equations result.

These formulae can be of practical use, both by themselves and in providing checks on numerical or analytical results. Some further products follow easily:

$$URZ = ZRU = \frac{a}{2} + \frac{a}{2} URN = \frac{a}{2}\alpha_0$$

$$UTZ = ZTU = \frac{a}{2} UTN = \frac{a}{2}\beta_0$$

$$NRZ = ZRN = \infty \text{ (diverges)}$$

$$NTZ = ZTN = \frac{a}{2} NTN = \frac{a}{4}\beta_{-1}$$

$$ZRZ = \infty \text{ (diverges)}$$

$$ZTZ = \frac{a^2}{4} NTN = \frac{a^2}{8}\beta_{-1}$$

It may be noted that the divergence of NRZ is caused only by the divergence of the first-order term NR_1Z (compare Table 7). We can single this term out and thus derive a formula describing the behaviour of $X(\mu)$ near $\mu = 0$. After some reduction we find

$$Z(R - aR_1)N = \frac{a}{2}NRN = \frac{a}{4}\alpha_{-1}^{*} = \lim_{\mu \to 0}\left\{\frac{X(\mu) - 1}{2\mu} + \frac{a}{4}\ln\mu\right\}$$

It is thus found that $X'(\ ) = \frac{dX(\mu)}{d\mu}$ diverges logarithmically at $\mu = 0$.

However $Y'(\mu) = \frac{dY(\mu)}{d\mu}$ attains a finite limit by the equation

$$ZTN = \frac{a}{2} NTN = \frac{a}{4} \beta_{-1} = \lim_{\mu \to 0} \frac{Y(\mu)}{2\mu} = \frac{1}{2} Y'(0)$$

9. The Method of Successive Order Scattering (Simple Iteration)

Suppose we wish to have the numerical value of the reflected radiation for a given incident radiation field I, a given detector characteristic D, a given optical thickness, b, and a given albedo a. The answers may be written in the form

$$DRI = DP\ C\ \tfrac{1}{4}\ a\ \bar{P}I =$$

$$= (DP)\ (1 + aM + a^2M^2 + a^3M^3 + \text{----})\ (\tfrac{1}{4}\ a\ \bar{P}I)$$

One obvious way of finding this result is to start with the τ-vector $S = \frac{1}{4}\ a\ \bar{P}I$, multiply with the $\tau\tau$-matrix aM many times in succession, take the sum, and multiply this sum with the τ-vector DP. The successive terms can be identified with successive order scattering, as explained in section 7.

This method has not usually been considered practical, except for very thin atmospheres, because the number of terms to be taken is quite large, whereas, for instance, in the non-linear integral equations for $X(\mu)$ and $Y(\mu)$ employed by Chandrasekhar a few

iterations suffice to give good accuracy. However, performing many simple operations and summing many terms is no problem for a fast computing machine. Some advantages of the simple iteration method are that all operations are linear and that the program is quite flexible, permitting any form for I or D to be put in. The conservative case (a = 0) does not require a special treatment, nor does the diffuse transmission for $\mu = \mu_0$ require a separate formula, as it does in the non-linear method. Finally, generalization of the method to arbitrary anisotropic scattering patterns is relatively simple.

There is only a minor difference between this method and the iteration method by means of a Neumann series (see Busbridge's book). In that method the equation

$$J = S + aMJ$$

is solved by the iteration procedure:

$$J_{(1)} = S$$

$$J_{(p+1)} = S + aMJ_{(p)}$$

$$J = \lim_{p \to \infty} J_{(p+1)}$$

In the present method we do not compute these successive approximations but the separate increments

$$a^p J_p = (aM)^{p-1} S = J_{(p)} - J_{(p-1)}$$

and take the sum later. This incidentally, makes it possible to put in different values of a.

An important practical modification is that after a certain term the sum may be replaced by the sum of a geometric series using the eigenvalues. The integral equation

$$\eta J = MJ$$

has a solution only for a discrete set of numbers $\eta^{(m)} < 1$ ($m = 1, 2, 3, \ldots$), the eigenvalues; the solutions are the corresponding eigenfunctions $J^{(m)}(\tau)$. A normalization convention is required to define them completely. There are no degeneracies here. Elaborate theories of the properties of eigenfunctions and of expansions into eigenfunctions exist, both for matrices M of discrete numbers and for continuous functions. This full theory does not have to be invoked, for the application to the present computation is quite simple. If any term of the series for J can be represented with good accuracy as a linear combination of two eigenfunctions

$$a^p J_p = a^p \left\{ q_1 J^{(1)} + q_2 J^{(2)} \right\}$$

then the next term will be

$$a^{p+1} J_{p+1} = a^{p+1} M J_p = a^{p+1} \left\{ \eta^{(1)} q_1 J^{(1)} + \eta^{(2)} q_2 J^{(2)} \right\}$$

The numerical value of the term with the lower eigenvalue, say $\eta^{(2)}$, thus drops more rapidly than the term with the larger eigenvalue $\eta^{(1)}$. Eventually, there will be a term beyond which we can neglect with good accuracy all but the lowest eigenfunction and put

$$J_{p+1} = a \, \eta^{(1)} J_p \qquad p \geq \tau$$

From this point on we may sum in a geometrical series

$$\sum_{p=\tau}^{\infty} a^p J_p = \frac{a^r J_r}{1 - a \eta^{(1)}}$$

Since $a \leq 1$ and $\eta^{(1)} < 1$ the convergence is assured. Only the conservative case ($a = 1$) for a semi-infinite atmosphere ($\eta^{(1)} = 1$) has to be treated as an exception. This is exactly the case which has been thoroughly discussed by Hopf and others a long time ago. If albedo values $a > 1$ were permitted, a "critical" value $a = (\eta^{(1)})^{-1}$ would be reached at which the radiation in the slab is selfsustained. This corresponds to the critical state of a nuclear reactor.

We have found no values for $\eta^{(1)}$ in the literature, but

estimated some in an earlier paper (Van de Hulst and Irvine, paper at Astrophysical Symposium, Liege, 1962). Some more accurate values now available are presented in Table 9.

The values marked "numerical" in Table 9 have been obtained from the iteration process. Those for $\eta^{(3)}$ followed from the way in which the ratios WJ_{p+1}/WJ_p approached their limit $\eta^{(1)}$. Here W = operator defining integration over τ from 0 to b. The values marked "approximate" in Table 9 came from two sets of formulae derived from small and large b. A horizontal line separates the values derived from these two sets.

For small b:

$$\eta^{(0)} = \frac{1}{2b}\{2E_3(b) - 1 + 2b\}$$

$$\eta^{(2)} = \frac{1}{b}\{4E_3(\tfrac{1}{2}b) - E_3(b) - \tfrac{3}{2} + b\}$$

These formulae follow from assuming a simple rectangular source function or a function which is 1 for $0 < \tau < \frac{1}{2}b$ and -1 for $\frac{1}{2}b < \tau < b$ and by applying the Milne operator once and taking the integral. The approximation seems quite satisfactory for $b \leqslant 0.5$.

For large b:

$$\frac{1}{\eta^{(m)}} - 1 = \frac{m^2\pi^2}{3(b+1.40)^2} \qquad (m = 1, 2, 3 \ldots)$$

TABLE 9: Eigenvalues for Isotropic Scattering

b	numerical	approx.	approx.	numerical	approx.
0.01	.0276	.0276			
0.02		.0484	.0069		
0.05		.0984			
0.1	.164	.1630	.0338		
0.2		.260	.066		
0.4		.393	.126		
0.5	.44734	.443		.0845	
0.8		.555	.231		
1.0	.61902	.610	.306	.195 ± .005	.164
2.0	.78301	.779	.449	.345 ± .005	.281
4	.90213	.899	.691	.565 ± .005	.498
8		.964	.871		.750
10	.97549	.976	.909	.829 ± .002	.816
20		.9929	.972		.939
∞	1	1	1	1	1

This formula, which gives a satisfactory approximation for m = 1, $b \geqslant 2$, has been derived by assuming for J a sine curve

$$J(\tau) = \sin \frac{m\pi(\tau + 0.70)}{b + 1.40}$$

which has (m - 1) zeros inside the atmosphere and would reach zero outside the atmosphere at $\tau = -0.70$ and $\tau = b + 0.70$. The value 0.70 is somewhat ill-defined but not very critical; it can be estimated either from the theory for a semi-infinite atmosphere or from numerically computed eigenfunctions. We now combine the equation

$$\frac{dF(\tau)}{d\tau} = 4\left(1 - \frac{1}{a}\right) J(\tau)$$

derived on page 17, with the equation

$$F(\tau) = \frac{4}{3} \frac{dJ(\tau)}{d\tau}$$

which is a well known approximation valid in any point inside a thick atmosphere not too close to the surface. These equations are consistent if we choose for a the critical value > 1 defined by

$$1 - \eta = 1 - \frac{1}{a} = \frac{m^2\pi^2}{3(b + 1.40)^2}$$

This result cannot be expected to be correct, except for very large values of b. The formula actually used is equivalent for

large b, but fits the data better for relatively small b.

The eigenvalues in Table 9 can be used for an estimate of the number of iterations required. One-sided illumination generally generates a first order source function $S(\tau)$ in which the second eigenfunction $J^{(2)}(\tau)$ is represented with a coefficient of order unity. Hence 6-figure accuracy can be obtained if the iteration is stopped (and further terms replaced by a geometric series) when this term has been reduced to 10^{-6}, i.e., at the order

$$\tau = \frac{6}{-\log \eta^{(2)}}$$

An initially symmetric source distribution gives faster convergence because $J^{(3)}(\tau)$ is the next competing eigenfunction. Some typical estimates are given in Table 10. These data are not sufficiently complete for a critical discussion but show the general trend.

TABLE 10: Number of Iterations Required

b	one-sided illumination		symmetric sources	
	$6/(-\log \eta^{(2)})$	empirical r	$6/(-\log \eta^{(3)})$	empirical r
0.5	7	11	6	5
1	12	17	8	7
2	17		13	10
4	38		24	17
10	150		75	48

10. Adding method

By the adding method we shall understand a method for computing the reflection and transmission by a layer of optical thickness $b = b' + b''$ making use of the known reflection and transmission by layers of optical thickness b' and b''.

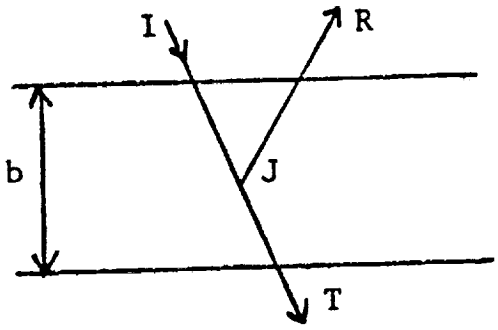
We shall derive the formulae in two independent ways. In the first derivation we use the external relations for the two composing layers and just look how the radiation can get out of the combined layer. This derivation holds quite generally, even for inhomogeneous layers and for anisotropic single scattering diagrams. The second derivation is more formal and makes use of partitioned matrices. As this derivation uses the radiation densities and the albedo explicitly, its validity is confined to homogeneous layers with isotropic scattering and constant albedo.

Our main aim in discussing the adding method is to apply it to the case $b' = b''$, i.e. in the form of a doubling method. In another extreme we might take either b' or b'' infinitesimal. This leads to the method applied by Bellmann, Kalaba and Prestrud and described by them as invariant embedding.

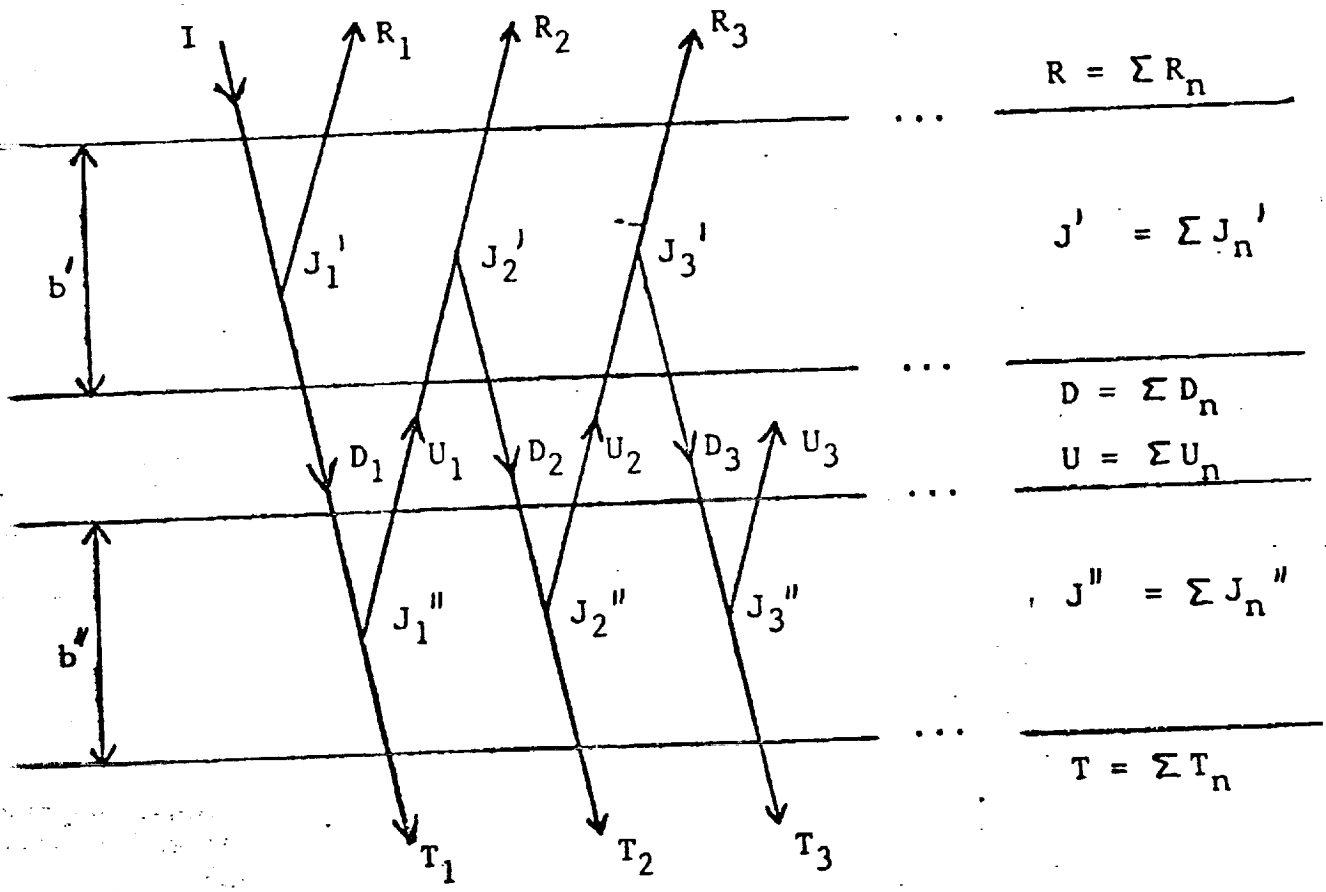
For the first derivation we refer to Fig. 2. In the prototype diagram I is the incident, R the reflected and T the transmitted radiation, and J the radiation density (which will be used only in the second derivation). The R, T, and J, include all orders of scattering and thus form the exact solution for the equation of transfer in the single layer. The

Figure 2. The doubling method

(a) proto-type, single layer



(b) double layer



T includes the zero-order term corresponding to direct transmission. The double-layer diagram consists of a simple succession of these prototype diagrams, namely in the upper half one right side up and the rest upside down, and in the lower half all right side up. The radiation intensities up and down at the center of the double layer are called U and D. Primes refer to the upper layer, double primes to the lower layer. Note that the indices n in this method do not signify orders of scattering; the meaning of n - 1 is (in R_n, T_n, J'_n and J''_n) the number of times a photon has crossed the middle boundary going up.

The recursion formulae can be read from Figure 2 and are for incident radiation from one direction

$$R_1 = R' \qquad D_1 = T'$$

$$T_n = T'' D_n \qquad U_n = R'' D_n$$

$$R_{n+1} = T' U_n \qquad D_{n+1} = R' U_n \qquad n = 1, 2, \ldots$$

By addition they give

$$D = \left\{1 + (R'R'') + (R'R'')^2 + \ldots\right\} T' = (1 - R'R'')^{-1} T'$$

$$U = R'' D$$

$$R = R' + T'R'' D$$

$$T = T'' D$$

These equations are correct and complete but deceivingly simple. Great care has to be exercised in writing them in the forms of integrals, because the matrix D contains, like T', T'' and T, a singular (diagonal) part besides a diffuse part. Also, the order of arguments is not irrelevant in the assymetric matrices D and U. For further explanation we write therefore the same set of equations in a form in which they can be used for actual numerical calculation.

Write positive cosines of angles with the normal

for incident radiation at top: μ_0

for downward radiation at interface: u, v, w

for upward radiation at interface: z

for emergent radiation at top or bottom: μ

The calculation may then proceed as shown in Table 11.

Let us now turn to the second more formal derivation. We refer to the book of V. N. Faddeeva "Computational methods of linear algebra" and use a combination of the method of

TABLE 11: Computation Program for Adding Method

Matrix form	Functional form
$Q_1 = R' R''$	$Q_1(u,v) = \int_0^1 R'(u,z) R''(z,v)\, 2z\, dz$
$Q_{n+1} = Q_1 Q_n$	$Q_{n+1}(u,v) = \int_0^1 Q_1(u,w) Q_n(w,v)\, 2w\, dw$
$S = \sum_{n=1}^{\infty} Q_n$	$S(u,v) = \sum_{n=1}^{\infty} Q_n(u,v)$
$D = (1+S)(T_0' + T_{diff}')$	$D_{diff}(u,\mu_0) = T_{diff}'(u,\mu_0) + S(u,\mu_0)\, e^{-b'/\mu_0}$
$= T_0' + D_{diff}$	$+ \int_0^1 S(u,v)\, T_{diff}'(v,\mu_0)\, 2v\, dv$
$U = R''(T_0' + D_{diff})$	$U(z,\mu_0) = R''(z,\mu_0)\, e^{-b'/\mu_0}$
	$+ \int_0^1 R''(z,u)\, D_{diff}(u,\mu_0)\, 2u\, du$
$R = R' + (T_0' + T_{diff}')\, U$	$R(\mu,\mu_0) = R'(\mu,\mu_0) + e^{-b/\mu}\, U(\mu,\mu_0)$
	$+ \int_0^1 T_{diff}'(\mu,z)\, U(z,\mu_0)\, 2z\, dz$
$T = (T_0'' + T_{diff}'')(T_0' + D_{diff})$	$T_{diff}(\mu,\mu_0) = e^{-b''/\mu} D_{diff}(\mu,\mu_0') + T_{diff}''(\mu,\mu_0)\, e^{-b'/\mu_0}$
$= T_0 + T_{diff}$	$+ \int_0^1 T_{diff}''(\mu,u)\, D_{diff}(u,\mu_0)\, 2u\, du$

partitioned matrices (p. 102 - 103) with that of improved convergence of iteration (p. 127 - 131).

We write vectors and matrices for the double layer without primes, vectors and matrices for the separate layers with primes and double primes. The central problem (see section 5) is to find

$$C = (1 - aM)^{-1}$$

when M and a are given. Suppose we know the relations

$$C' = (1 - aM')^{-1}$$

$$C'' = (1 - aM'')^{-1}$$

We now partition the matrices for the composite layer and introduce a new one, G, as follows

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M = \begin{pmatrix} M' & \bar{L} \\ L & M'' \end{pmatrix}$$

$$G = \begin{pmatrix} C' & o \\ o & C'' \end{pmatrix}$$

Obviously G can serve as a first approximation to C. We find by direct multiplication

$$G(1 - aM) = 1 - H$$

where

$$H = \begin{pmatrix} o & aC'\bar{L} \\ aC''L & o \end{pmatrix}$$

and hence

$$(1 - H)C = G$$

or

$$C = (1 - H)^{-1}G$$

Inversion of 1 - H in partitioned form gives by standard procedures

$$(1 - H)^{-1} = \begin{Bmatrix} (1 - a^2C'\bar{L}C''L)^{-1} & aC'L(1 - a^2C''LC'\bar{L})^{-1} \\ aC''L(1 - a^2C'\bar{L}C''L)^{-1} & (1 - a^2C''LC'\bar{L})^{-1} \end{Bmatrix}$$

and finally,

$$C = \begin{Bmatrix} (1 - a^2C'\bar{L}C''L)^{-1}C' & (1 - a^2C'\bar{L}C''L)^{-1}aC'\bar{L}C'' \\ (1 - a^2C''LC'\bar{L})^{-1}aC''LC' & (1 - a^2C''LC'\bar{L})^{-1}C'' \end{Bmatrix}$$

The method thus followed consists of choosing a trial solution G which is used to improve the convergence of the iteration. For large values of b (thick layers) and albedo 1 or close to 1, the convergence of the iteration procedure by successive orders of scattering, i.e. successive multiplication by aM, is extremely slow. The expectation is that the iterations involved in the equations just derived, for instance,

$$(1 - a^2C'\bar{L}C''L)^{-1} = 1 + a^2C'\bar{L}C''L + a^4C'\bar{L}C''LC'\bar{L}C''L + \ldots$$

converge more rapidly. We shall later confirm this by looking at the eigen values.

We still have to establish the equivalence between the results of the second derivation (expressed in terms of $\tau'\tau'$, $\tau'\tau''$, $\tau''\tau'$, and $\tau''\tau''$ matrices) and of the first derivation (expressed in terms of $\mu\mu$ matrices). This

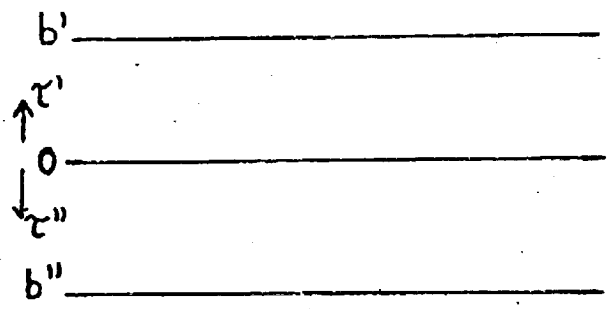
equivalence is demonstrated by considering that a geometric series in which the ratio of successive terms is a product, say AB, can be written in different ways namely

$$(1 - AB)^{-1} = 1 + AB + ABAB + \ldots =$$

$$1 + A(1 + BA + BABA + \ldots)B = 1 + A(1 - BA)^{-1} B$$

This transformation is trivial if A and B are numbers, but non-trivial for matrices. Generally A can be a matrix of n rows and m columns and B one of m rows and n columns; then AB is a square matrix of n x n components and BA a square matrix of m x m components.

The equivalence is now established as follows. Let us measure τ' from the separation layer up in the top layer to b′ and τ'' down in the bottom layer to b″ (see Figure 3).



(Figure 3)

We have derived earlier that the matrices occurring in the first derivation can be decomposed as follows.

$$R' = P' C' \frac{a}{4} \bar{P}'$$

$$(\mu\mu) \quad (\mu\tau')(\tau'\tau')(\tau'\mu)$$

and

$$R'' = P'' C'' \frac{a}{4} \bar{P}''$$

$$(\mu\mu) \quad (\mu\tau'')(\tau''\tau'')(\tau''\mu)$$

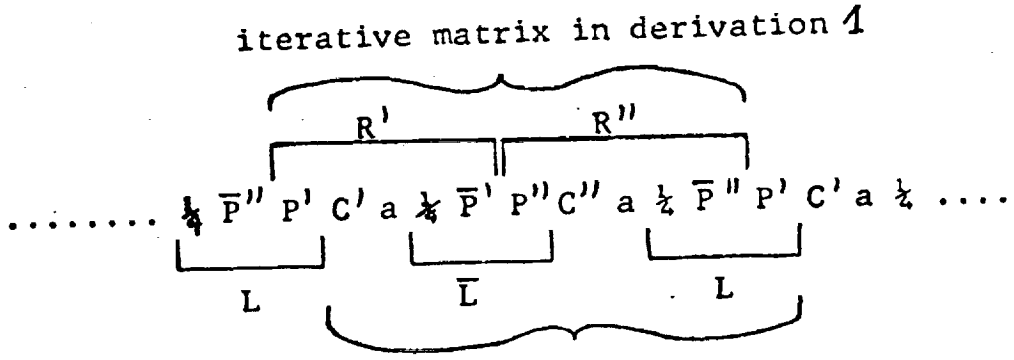
The matrices L and $\bar{L}$ recurring in the second derivation have the functional form

$$L = \bar{L} = \tfrac{1}{2}E_1(\tau' + \tau'')$$

and can also be written as matrix products:

$$L = \tfrac{1}{4}\,\bar{P}''P' \qquad\qquad \bar{L} = \tfrac{1}{4}\,\bar{P}'P''$$

We thus find that the general term of the iterative solution contains continued products as follows



The remaining proof consists of simple substitutions. We generally have

$$R = P\,C\,S$$

$$T = T_o + Q\,C\,S$$

where $T_o = T_o' T_o''$ and, considering the convention of counting τ' and τ'' (Figure 2), the partitioned forms are

$$S = \begin{pmatrix} S' \\ S'' \end{pmatrix} = \begin{pmatrix} \frac{a}{4}\,\bar{Q}' \\ \frac{a}{4}\,\bar{P}''\,T_o' \end{pmatrix} \qquad P = \begin{pmatrix} Q' \\ T_o'\,P'' \end{pmatrix} \qquad Q = \begin{pmatrix} T_o''\,P' \\ Q'' \end{pmatrix}$$

The partitioned form of C was found in derivation 2. Performing

the somewhat tedious multiplications we find indeed the same result as when writing out the formulae of derivation 1, namely

$$R = R' + T'(R'' + \ldots)\, T' =$$

$$= Q'C'a\tfrac{1}{4}\bar{Q}' + (T_o' + Q'C'a\tfrac{1}{4}\bar{P}')(P''C''a\tfrac{1}{4}\bar{P}'' + \ldots)(T_o' + P'C'a\tfrac{1}{4}\bar{Q}')$$

and

$$T = T''(1 + \ldots)\, T' =$$

$$= (T_o'' + Q''C''a\,\tfrac{1}{4}\,\bar{P}'')(1 + \ldots)(T_o' + P'C'\,a\,\tfrac{1}{4}\,Q')$$

which follow from the first derivation. The dots stand for higher powers in $R''R'$ or $R'R''$.

A sample computation was made starting with $R(\mu, \mu_o)$ and $T_{diff}(\mu, \mu_o)$ for b = 0.25 obtained by the simple iteration method and proceeding to find these functions for b = 0.5, 1, 2, 4, 8 and 16, by the doubling method. Only μ-integrations are involved. In spite of the fact that a fairly crude integration scheme was used, three to four-figure accuracy was obtained.

It is interesting to examine the eigenvalues $\alpha(b,a)$ found in this computation. For given b and a the ratio of successive terms in the doubling method approaches a definite limit:

$$\alpha(b,a) = \lim_{n\to\infty} \frac{Q_{n+1}(b,a,u,v)}{Q_n(b,a,u,v)}$$

In all sample computations the ratio came to within one per cent of this limit already at the iteration from Q_2 to Q_3. This shows that the next highest eigenvalue is quite low and that without loss of accuracy the sum might have been replaced by a geometric series beyond the third or fourth terms. This is entirely different from what occurs in the simple iteration procedure. Also, the eigenvalues α are much lower anyway than the eigenvalues η of the simple iteration procedure.

For thick layers the radiation field at the central interface must be almost isotropic. Hence we should expect the eigenvalue α to approach the square of

$$r_c\,(b/2) = URU$$

because at both reflections R' and R" this is the fraction of the flux thrown back across the interface. Table 12 shows that this expectation is well fulfilled.

TABLE 12: Eigenvalues in the Doubling Method

½b	b	a=0.8 α	a=0.8 $(URU)^2$	a=0.95 α	a = 1.00 α	a = 1.00 $(URU)^2$
0.25	0.5	.0267	.0176	.0410	.0472	.0324
0.5	1	.0520	.0422	.0892	.1066	.087
1	2	.0870	.0785	.172	.2172	.200
2	4	.114		.272	.3838	
4	8	.122		.343	.4761	
8	16	.122		.361	.743	

II. Some Numerical Results

Table 13 (pages 73 to 76) presents some sample pages of a numerical table which has been designed to give maximum convenience to the "user". The user may be a physicist, astronomer, geophysicist, or other scientist who wishes to make rapid reference to a particular number or who wishes to employ such a number for comparison with a more complicated problem he wishes to study. The headings are in physical rather than mathematical terms, but the mathematical definitions of the given numbers are:

for reflection (out at top)	DR_nI	(n = 1, 2, 3)
	DRI	(8 values of a)
for transmission (out at bottom)	DT_nI	(n = 1, 2, 3, 4)
	DTI	(8 values of a)

There are eight choices of incident radiation field I, corresponding to six separate angles of incidence and two distributed fields, N and U. Similarly there are nine choices of the "detector" D of the emerging radiation corresponding to the same eight choices and the grazing emergence, vector Z, $\mu = 0$. The direct transmission has a singularity if D and I refer to the same angle. This has been indicated by the word PEAK in the appropriate boxes. The DTI values in the columns under the word

PEAK refer to the diffuse transmission only. The DTI values in all other columns include the direct radiation DT_oI which has no singularity there. The 72 reflection columns and, similarly, the 72 transmission columns are partly redundant. Because of the reciprocity between I and D twenty-eight columns occur twice and sixteen columns, namely the eight symmetric combinations and the eight combinations with D = Z, occur once.

The computations were programmed by M. K. Grossman and carried out at the IBM 7090 of the Institute for Space Studies, New York. The simple iteration method (successive scattering) was used. Abundant internal checks on the accuracy are provided by the required equality of the twenty-eight reciprocal products. Generally, the present integration, which was carried out with steps 0.01 in τ, gave an accuracy of about four units in the fifth decimal. Only the functions involving the directions $\mu_o = 0.1$ or $\mu = 0.1$ gave differences up to twenty-six units in the fifth decimal. This may be remedied, if desired, by a finer integration mesh. Additional internal checks are available between the products with vector Z and those with vector N (section 8).

The functions $X(\mu)$ and $Y(\mu)$ and their moments of order -1, 0 and 1 are implicitly contained in these tables in various combinations (page 36-38). The checks against other published

ISOTROPIC SCATTERING FINITE LAYER TABLE 13 FOR EXPLANATION SEE PAGE 71-72
INTENSITIES OUT AT TOP

VECTOR	MU=0 Z	MU=0.1	MU=0.3	MU=0.5	MU=0.7	MU=0.9	MU=1.0	AVERAGE N	FLUX U
B= 1.0 NARROW SOURCE LAYER AT TOP									
FIRST ORDER	00.00000	0.59997	0.36529	0.27050	0.21553	0.17924	0.16534	0.34270	0.24458
SECOND ORDER	0.17135	0.18265	0.15384	0.12741	0.10728	0.09216	0.08601	0.13227	0.11390
THIRD ORDER	0.06614	0.08188	0.08018	0.07022	0.06075	0.05301	0.04975	0.06877	0.06284
SUMS A= 0.20	20.00745	0.12803	0.07994	0.05983	0.04795	0.04002	0.03696	0.07445	0.05404
A= 0.40	40.03279	0.27591	0.17741	0.13447	0.10849	0.09091	0.08409	0.16398	0.12133
A= 0.60	60.08283	0.45217	0.30116	0.23174	0.18845	0.15868	0.14704	0.27611	0.20887
A= 0.80	80.17015	0.67279	0.46802	0.36682	0.30116	0.25506	0.23684	0.42538	0.33028
A= 0.90	90.23523	0.80921	0.57777	0.45777	0.37794	0.32117	0.29859	0.52273	0.41198
A= 0.95	95.27543	0.88691	0.64244	0.51204	0.42403	0.36099	0.33583	0.57986	0.46072
A= 0.99	99.31224	0.95486	0.70021	0.56090	0.46569	0.39706	0.36958	0.63080	0.50460
A= 1.00	00.32218	0.97277	0.71562	0.57399	0.47687	0.40675	0.37866	0.64437	0.51635
B= 1.0 MUNOUGHT = 0.1									
FIRST ORDER	2.50000	1.25004	0.62500	0.41667	0.31250	0.25000	0.22727	0.59997	0.38016
SECOND ORDER	0.29993	0.29983	0.21173	0.16201	0.13085	0.10961	0.10135	0.18266	0.14475
THIRD ORDER	0.09133	0.10912	0.09618	0.08046	0.06802	0.05855	0.05468	0.08192	0.07181
SUMS A= 0.20	0.51280	0.26297	0.13433	0.09054	0.06835	0.05491	0.05000	0.12803	0.08247
A= 0.40	1.05518	0.55667	0.29174	0.19924	0.15160	0.12243	0.11170	0.27592	0.18117
A= 0.60	1.63566	0.89156	0.48220	0.33476	0.25718	0.20901	0.19115	0.45219	0.30383
A= 0.80	2.26913	1.28628	0.72430	0.51351	0.39938	0.32716	0.30009	0.67283	0.46515
A= 0.90	2.61418	1.51671	0.87573	0.62897	0.49284	0.40565	0.37276	0.80928	0.56917
A= 0.95	2.79632	1.64343	0.96246	0.69631	0.54789	0.45216	0.41591	0.88699	0.62979
A= 0.99	2.94770	1.75166	1.03855	0.75610	0.59706	0.49385	0.45464	0.95496	0.68359
A= 1.00	2.98644	1.77980	1.05865	0.77200	0.61018	0.50500	0.46500	0.97287	0.69789
B= 1.0 MUNOUGHT = 0.3									
FIRST ORDER	0.83333	0.62500	0.41614	0.31099	0.24786	0.20589	0.18978	0.36529	0.27821
SECOND ORDER	0.18265	0.21158	0.18079	0.14907	0.12503	0.10712	0.09987	0.15379	0.13290
THIRD ORDER	0.07690	0.09612	0.09380	0.08174	0.07051	0.06142	0.05760	0.08017	0.07307
SUMS A= 0.20	0.17466	0.13433	0.09131	0.06890	0.05521	0.04602	0.04248	0.07994	0.06162
A= 0.40	0.36881	0.29171	0.20317	0.15505	0.12508	0.10467	0.09675	0.17740	0.13867
A= 0.60	0.59034	0.48212	0.34579	0.26763	0.21749	0.18287	0.16933	0.30114	0.23927
A= 0.80	0.85386	0.72414	0.53865	0.42408	0.34785	0.29417	0.27297	0.46798	0.37916
A= 0.90	1.00998	0.87551	0.66568	0.52945	0.43666	0.37052	0.34423	0.57772	0.47342
A= 0.95	1.09680	0.96220	0.74054	0.59230	0.48995	0.41650	0.38721	0.64233	0.52966
A= 0.99	1.17157	1.03826	0.80743	0.64887	0.53811	0.45814	0.42617	0.70014	0.58030
A= 1.00	1.19111	1.05834	0.82527	0.66403	0.55104	0.46933	0.43664	0.71555	0.59387
B= 1.0 MUNOUGHT = 0.5									
FIRST ORDER	0.50000	0.41667	0.31099	0.24542	0.20158	0.17062	0.15837	0.27050	0.21947
SECOND ORDER	0.13525	0.16188	0.14906	0.12719	0.10861	0.09406	0.08803	0.12738	0.11364
THIRD ORDER	0.06369	0.08041	0.08174	0.07259	0.06321	0.05537	0.05203	0.07021	0.06500
SUMS A= 0.20	0.10598	0.09053	0.06890	0.05483	0.04523	0.03839	0.03567	0.05983	0.04903
A= 0.40	0.22689	0.19922	0.15509	0.12463	0.10334	0.08798	0.08183	0.13447	0.11144
A= 0.60	0.36952	0.33469	0.26763	0.21756	0.18149	0.15506	0.14442	0.23173	0.19457
A= 0.80	0.54671	0.51337	0.42407	0.34957	0.29371	0.25203	0.23510	0.36678	0.31270
A= 0.90	0.65598	0.62877	0.52944	0.43998	0.37121	0.31932	0.29814	0.45773	0.39366
A= 0.95	0.71819	0.69609	0.59228	0.49438	0.41804	0.36008	0.33636	0.51199	0.44239
A= 0.99	0.77262	0.75585	0.64885	0.54362	0.46055	0.39714	0.37113	0.56014	0.48650
A= 1.00	0.78697	0.77173	0.66401	0.55685	0.47198	0.40712	0.38049	0.57393	0.49836

ISOTROPIC SCATTERING FINITE LAYER TABLE 13
INTENSITIES OUT AT TOP

FOR EXPLANATION SEE PAGE 71-72

VECTOR	MU=0 Z	MU=0.1	MU=0.3	MU=0.5	MU=0.7	MU=0.9	MU=1.0	AVERAGE N	FLUX U
B= 1.0 MUNOUGHT = 0.7									
FIRST ORDER	0.35714	0.31250	0.24786	0.20158	0.16832	0.14392	0.13409	0.21553	0.18048
SECOND ORDER	0.10777	0.13074	0.12502	0.10861	0.09361	0.08152	0.07644	0.10725	0.09718
THIRD ORDER	0.05363	0.06797	0.07050	0.06321	0.05531	0.04859	0.04571	0.06074	0.05666
SUMS A= 0.20	0.07622	0.06834	0.05521	0.04523	0.03791	0.03249	0.03029	0.04795	0.04050
A= 0.40	0.16455	0.15158	0.12507	0.10334	0.08698	0.07473	0.06974	0.10848	0.09252
A= 0.60	0.27082	0.25713	0.21749	0.18149	0.15351	0.13227	0.12357	0.18844	0.16250
A= 0.80	0.40617	0.39926	0.34784	0.29371	0.24989	0.21605	0.20210	0.30113	0.26305
A= 0.90	0.49149	0.49268	0.43664	0.37121	0.31689	0.27451	0.25696	0.37791	0.33251
A= 0.95	0.54068	0.54769	0.48993	0.41804	0.35751	0.31003	0.29032	0.42399	0.37450
A= 0.99	0.58406	0.59684	0.53809	0.46054	0.39446	0.34237	0.32070	0.46564	0.41262
A= 1.00	0.59555	0.60995	0.55102	0.47198	0.40441	0.35109	0.32889	0.47682	0.42287
B= 1.0 MUNOUGHT = 0.9									
FIRST ORDER	0.27778	0.25000	0.20589	0.17062	0.14392	0.12384	0.11564	0.17924	0.15293
SECOND ORDER	0.08962	0.10952	0.10712	0.09406	0.08152	0.07123	0.06687	0.09214	0.08425
THIRD ORDER	0.04607	0.05851	0.06141	0.05537	0.04859	0.04276	0.04025	0.05301	0.04967
SUMS A= 0.20	0.05956	0.05491	0.04602	0.03839	0.03249	0.02801	0.02617	0.04002	0.03441
A= 0.40	0.12929	0.12241	0.10467	0.08798	0.07473	0.06455	0.06037	0.09091	0.07885
A= 0.60	0.21427	0.20897	0.18287	0.15506	0.13227	0.11455	0.10721	0.15867	0.13899
A= 0.80	0.32424	0.32706	0.29416	0.25202	0.21605	0.18764	0.17581	0.25504	0.22593
A= 0.90	0.39451	0.40551	0.37050	0.31931	0.27451	0.23881	0.22387	0.32113	0.28629
A= 0.95	0.43534	0.45199	0.41648	0.36008	0.31003	0.26994	0.25314	0.36095	0.32287
A= 0.99	0.47152	0.49366	0.45812	0.39713	0.34237	0.29831	0.27981	0.39701	0.35611
A= 1.00	0.48113	0.50480	0.46932	0.40711	0.35108	0.30596	0.28701	0.40670	0.36507
B= 1.0 MUNOUGHT = 1.0									
FIRST ORDER	0.25000	0.22727	0.18978	0.15837	0.13409	0.11564	0.10808	0.16534	0.14202
SECOND ORDER	0.08267	0.10127	0.09986	0.08303	0.07644	0.06687	0.06281	0.08599	0.07888
THIRD ORDER	0.04299	0.05464	0.05759	0.05203	0.04571	0.04025	0.03789	0.04974	0.04668
SUMS A= 0.20	0.05370	0.05000	0.04247	0.03567	0.03029	0.02617	0.02447	0.03696	0.03198
A= 0.40	0.11682	0.11168	0.09675	0.08183	0.06974	0.06037	0.05650	0.08409	0.07338
A= 0.60	0.19411	0.19111	0.16933	0.14442	0.12357	0.10721	0.10042	0.14703	0.12951
A= 0.80	0.29473	0.30000	0.27296	0.23510	0.20210	0.17581	0.16482	0.23682	0.21084
A= 0.90	0.35935	0.37263	0.34422	0.29813	0.25696	0.22387	0.20999	0.29856	0.26740
A= 0.95	0.39700	0.41575	0.38720	0.33636	0.29031	0.25314	0.23751	0.33579	0.30171
A= 0.99	0.43042	0.45447	0.42615	0.37112	0.32070	0.27981	0.26260	0.36954	0.33291
A= 1.00	0.43931	0.46482	0.43662	0.38049	0.32889	0.28701	0.26937	0.37862	0.34131
B= 1.0 LAMBERT SURFACE ON TOP									
FIRST ORDER	0.50000	0.38016	0.27821	0.21947	0.18048	0.15293	0.14202	0.24458	0.19673
SECOND ORDER	0.12229	0.14466	0.13291	0.11365	0.09719	0.08426	0.07889	0.11388	0.10162
THIRD ORDER	0.05694	0.07176	0.07307	0.06500	0.05666	0.04967	0.04669	0.06284	0.05823
SUMS A= 0.20	0.10540	0.08247	0.06162	0.04903	0.04050	0.03441	0.03198	0.05404	0.04394
A= 0.40	0.22427	0.18115	0.13867	0.11145	0.09252	0.07885	0.07338	0.12133	0.09986
A= 0.60	0.36266	0.30378	0.23927	0.19457	0.16251	0.13899	0.12951	0.20886	0.17432
A= 0.80	0.53210	0.46505	0.37916	0.31271	0.26306	0.22594	0.21085	0.33026	0.28017
A= 0.90	0.63538	0.56902	0.47342	0.39367	0.33253	0.28630	0.26742	0.41196	0.35273
A= 0.95	0.69383	0.62962	0.52967	0.44240	0.37452	0.32288	0.30172	0.46069	0.39642
A= 0.99	0.74476	0.68340	0.58031	0.48652	0.41264	0.35613	0.33292	0.50456	0.43598
A= 1.00	0.75816	0.69769	0.59388	0.49838	0.42290	0.36509	0.34133	0.51632	0.44661

ISOTROPIC SCATTERING FINITE LAYER TABLE 13
INTENSITIES OUT AT BOTTOM

FOR EXPLANATION SEE PAGE 71-72

VECTOR	MU=0 Z	MU=0.1	MU=0.3	MU=0.5	MU=0.7	MU=0.9	MU=1.0	AVERAGE N	FLUX U
B= 1.0 NARROW SOURCE LAYER AT TOP									
ZERO ORDER	0.	0.00023	0.05946	0.13534	0.17118	0.18288	0.18394	0.10969	0.14850
FIRST ORDER	0.05485	0.06666	0.10349	0.11897	0.11814	0.11176	0.10793	0.10366	0.11225
SECOND ORDER	0.05183	0.06591	0.08547	0.08679	0.08097	0.07387	0.07043	0.07833	0.07947
THIRD ORDER	0.03917	0.05024	0.06031	0.05830	0.05300	0.04762	0.04516	0.05366	0.05283
SUMS A= 0.20	0.01341	0.01666	0.08413	0.16314	0.19853	0.20863	0.20876	0.13405	0.17461
A= 0.40	0.03366	0.04184	0.11975	0.20183	0.23594	0.24348	0.24225	0.16832	0.21065
A= 0.60	0.06570	0.08208	0.17368	0.25839	0.28968	0.29306	0.28972	0.21898	0.26292
A= 0.80	0.11966	0.15044	0.26085	0.34686	0.37234	0.36860	0.36178	0.29914	0.34410
A= 0.90	0.16133	0.20347	0.32632	0.41186	0.43238	0.42308	0.41363	0.35851	0.40345
A= 0.95	0.18794	0.23740	0.36756	0.45238	0.46960	0.45674	0.44563	0.39567	0.44036
A= 0.99	0.21290	0.26926	0.40594	0.48986	0.50391	0.48770	0.47505	0.43011	0.47445
A= 1.00	0.21974	0.27800	0.41640	0.50004	0.51321	0.49610	0.48302	0.43949	0.48371
B= 1.0 MUNOUGHT = 0.1									
ZERO ORDER	0.	Peak	0.	0.	0.	0.	0.	0.00023	0.00005
FIRST ORDER	0.00011	0.00113	0.04454	0.08456	0.09984	0.10286	0.10218	0.06666	0.08755
SECOND ORDER	0.03333	0.04115	0.06635	0.07622	0.07539	0.07108	0.06856	0.06594	0.07159
THIRD ORDER	0.03297	0.04203	0.05494	0.05579	0.05200	0.04741	0.04519	0.05027	0.05103
SUMS A= 0.20	0.00167	0.00227	0.01207	0.02048	0.02346	0.02385	0.02359	0.01666	0.02089
A= 0.40	0.00837	0.01086	0.03329	0.05088	0.05649	0.05660	0.05573	0.04185	0.05095
A= 0.60	0.02463	0.03157	0.07072	0.09810	0.10540	0.10392	0.10179	0.08211	0.09649
A= 0.80	0.06020	0.07689	0.13910	0.17627	0.18294	0.17727	0.17262	0.15049	0.17029
A= 0.90	0.09160	0.11697	0.19435	0.23589	0.24049	0.23086	0.22411	0.20355	0.22585
A= 0.95	0.11281	0.14408	0.23033	0.27373	0.27655	0.26420	0.25606	0.23749	0.26090
A= 0.99	0.13334	0.17033	0.26445	0.30910	0.31000	0.29499	0.28552	0.26937	0.29355
A= 1.00	0.13906	0.17764	0.27385	0.31876	0.31911	0.30336	0.29351	0.27811	0.30246
B= 1.0 MUNOUGHT = 0.3									
ZERO ORDER	0.	0.	Peak	0.	0.	0.	0.	0.05946	0.03567
FIRST ORDER	0.02973	0.04454	0.09909	0.12458	0.12749	0.12230	0.11864	0.1034[illegible]	[illegible].11850
SECOND ORDER	0.05174	0.06630	0.09218	0.09612	0.09067	0.08322	0.07951	0.085[illegible]	.08827
THIRD ORDER	0.04273	0.05490	0.06750	0.06592	0.06022	0.05425	0.05149	0.060[illegible]	.05981
SUMS A= 0.20	0.00841	0.01207	0.02413	0.02937	0.02968	0.02829	0.02738	0.08413	0.06345
A= 0.40	0.02395	0.03328	0.06026	0.07090	0.07068	0.06689	0.06459	0.11975	0.10235
A= 0.60	0.05210	0.07069	0.11672	0.13254	0.13017	0.12222	0.11770	0.17368	0.15954
A= 0.80	0.10434	0.13903	0.21050	0.23034	0.22259	0.20714	0.19889	0.26084	0.24952
A= 0.90	0.14683	0.19424	0.28215	0.30288	0.29014	0.26872	0.25758	0.32630	0.31588
A= 0.95	0.17458	0.23020	0.32764	0.34830	0.33215	0.30685	0.29388	0.36754	0.35732
A= 0.99	0.20093	0.26430	0.37016	0.39041	0.37095	0.34199	0.32729	0.40591	0.39569
A= 1.00	0.20819	0.27369	0.38179	0.40188	0.38148	0.35152	0.33635	0.41638	0.40612
B= 1.0 MUNOUGHT = 0.5									
ZERO ORDER	0.	0.	0.	Peak	0.	0.	0.	0.13534	0.13534
FIRST ORDER	0.06767	0.08456	0.12458	0.13534	0.13039	0.12116	0.11627	0.11897	0.12555
SECOND ORDER	0.05948	0.07616	0.09611	0.09549	0.08803	0.07973	0.07583	0.08678	0.08696
THIRD ORDER	0.04339	0.05575	0.06592	0.06306	0.05701	0.05106	0.04836	0.05829	0.05702
SUMS A= 0.20	0.01631	0.02047	0.02937	0.03146	0.03012	0.02789	0.02673	0.16314	0.16445
A= 0.40	0.04037	0.05087	0.07090	0.07483	0.07112	0.06559	0.06277	0.20183	0.20436
A= 0.60	0.07751	0.09807	0.13254	0.13758	0.12973	0.11909	0.11380	0.25838	0.26163
A= 0.80	0.13874	0.17619	0.23034	0.23475	0.21939	0.20037	0.19112	0.34684	0.35042
A= 0.90	0.18533	0.23577	0.30287	0.30563	0.28424	0.25886	0.24667	0.41184	0.41483
A= 0.95	0.21487	0.27359	0.34829	0.34967	0.32435	0.29495	0.28091	0.45235	0.45478
A= 0.99	0.24246	0.30893	0.39040	0.39030	0.36128	0.32813	0.31237	0.48982	0.49161
A= 1.00	0.25000	0.31859	0.40186	0.40133	0.37129	0.33712	0.32089	0.50001	0.50161

ISOTROPIC SCATTERING FINITE LAYER TABLE 13
INTENSITIES OUT AT BOTTOM

FOR EXPLANATION SEE PAGE 71-72

VECTOR	MU=0 Z	MU=0.1	MU=0.3	MU=0.5	MU=0.7	MU=0.9	MU=1.0	AVERAGE N	FLUX U
B= 1.0 MUNOUGHT = 0.7									
ZERO ORDER	0.	0.	0.	0.	Peak	0.	0.	0.17118	0.23965
FIRST ORDER	0.08559	0.09984	0.12749	0.13039	0.12227	0.11193	0.10686	0.11814	0.11982
SECOND ORDER	0.05907	0.07533	0.09067	0.08803	0.08024	0.07221	0.06852	0.08095	0.07987
THIRD ORDER	0.04048	0.05196	0.06021	0.05701	0.05128	0.04578	0.04332	0.05299	0.05147
SUMS A= 0.20	0.01985	0.02346	0.02968	0.03012	0.02813	0.02569	0.02451	0.19853	0.26728
A= 0.40	0.04719	0.05648	0.07068	0.07112	0.06613	0.06023	0.05740	0.23593	0.30476
A= 0.60	0.08690	0.10536	0.13017	0.12973	0.12001	0.10898	0.10375	0.28967	0.35816
A= 0.80	0.14893	0.18286	0.22258	0.21939	0.20177	0.18260	0.17361	0.37233	0.43958
A= 0.90	0.19456	0.24038	0.29013	0.28424	0.26057	0.23536	0.22362	0.43236	0.49833
A= 0.95	0.22305	0.27641	0.33214	0.32435	0.29683	0.26784	0.25440	0.46957	0.53464
A= 0.99	0.24942	0.30984	0.37093	0.36127	0.33016	0.29766	0.28264	0.50387	0.56804
A= 1.00	0.25659	0.31894	0.38147	0.37128	0.33919	0.30574	0.29028	0.51318	0.57709
B= 1.0 MUNOUGHT = 0.9									
ZERO ORDER	0.	0.	0.	0.	0.	Peak	0.	0.18288	0.32919
FIRST ORDER	0.09144	0.10286	0.12230	0.12116	0.11193	0.10160	0.09672	0.11176	0.11081
SECOND ORDER	0.05588	0.07102	0.08321	0.07973	0.07221	0.06475	0.06136	0.07385	0.07220
THIRD ORDER	0.03693	0.04737	0.05425	0.05105	0.04578	0.04080	0.03858	0.04761	0.04605
SUMS A= 0.20	0.02086	0.02385	0.02829	0.02789	0.02569	0.02328	0.02215	0.20863	0.35466
A= 0.40	0.04870	0.05659	0.06688	0.06559	0.06023	0.05448	0.05179	0.24348	0.38901
A= 0.60	0.08792	0.10389	0.12221	0.11909	0.10898	0.09836	0.09343	0.29306	0.43762
A= 0.80	0.14743	0.17719	0.20713	0.20037	0.18260	0.16438	0.15601	0.36858	0.51127
A= 0.90	0.19037	0.23076	0.26870	0.25886	0.23536	0.21158	0.20069	0.42305	0.56419
A= 0.95	0.21694	0.26407	0.30684	0.29495	0.26784	0.24050	0.22816	0.45671	0.59681
A= 0.99	0.24140	0.29485	0.34197	0.32812	0.29766	0.26723	0.25335	0.48767	0.62678
A= 1.00	0.24803	0.30321	0.35150	0.33711	0.30574	0.27443	0.26017	0.49606	0.63490
B= 1.0 MUNOUGHT = 1.0									
ZERO ORDER	0.	0.	0.	0.	0.	0.	Peak	0.18394	0.36788
FIRST ORDER	0.09197	0.10218	0.11864	0.11627	0.10686	0.09672	0.09197	0.10793	0.10617
SECOND ORDER	0.05397	0.06851	0.07950	0.07583	0.06852	0.06136	0.05812	0.07042	0.06862
THIRD ORDER	0.03521	0.04516	0.05149	0.04836	0.04332	0.03858	0.03647	0.04515	0.04361
SUMS A= 0.20	0.02088	0.02359	0.02738	0.02673	0.02451	0.02215	0.02105	0.20876	0.39226
A= 0.40	0.04845	0.05571	0.06459	0.06277	0.05740	0.05179	0.04920	0.24224	0.42505
A= 0.60	0.08691	0.10176	0.11770	0.11380	0.10375	0.09343	0.08868	0.28971	0.47137
A= 0.80	0.14471	0.17255	0.19888	0.19112	0.17361	0.15601	0.14796	0.36176	0.54140
A= 0.90	0.18612	0.22401	0.25757	0.24667	0.22362	0.20069	0.19026	0.41361	0.59162
A= 0.95	0.21166	0.25594	0.29387	0.28091	0.25440	0.22816	0.21624	0.44561	0.62256
A= 0.99	0.23513	0.28538	0.32728	0.31237	0.28263	0.25335	0.24007	0.47501	0.65096
A= 1.00	0.24149	0.29337	0.33634	0.32089	0.29028	0.26017	0.24651	0.48298	0.65865
B= 1.0 LAMBERT SURFACE ON TOP									
ZERO ORDER	0.	0.00005	0.03567	0.13534	0.23965	0.32919	0.36788	0.14850	0.21938
FIRST ORDER	0.07425	0.08755	0.11850	0.12555	0.11982	0.11081	0.10617	0.11225	0.11624
SECOND ORDER	0.05612	0.07154	0.08327	0.08697	0.07988	0.07221	0.06863	0.07946	0.07913
THIRD ORDER	0.03973	0.05100	0.05981	0.05702	0.05148	0.04606	0.04361	0.05282	0.05155
SUMS A= 0.20	0.01746	0.02089	0.06345	0.16445	0.26728	0.35467	0.39226	0.17461	0.24627
A= 0.40	0.04213	0.05094	0.10235	0.20436	0.30477	0.38901	0.42505	0.21065	0.28296
A= 0.60	0.07888	0.09646	0.15954	0.26184	0.35816	0.43762	0.47138	0.26292	0.33555
A= 0.80	0.13763	0.17022	0.24952	0.35043	0.43959	0.51128	0.54140	0.34409	0.41625
A= 0.90	0.18154	0.22574	0.31587	0.41484	0.49834	0.56420	0.59163	0.40343	0.47475
A= 0.95	0.20916	0.26078	0.35732	0.45479	0.53465	0.59682	0.62257	0.44034	0.51098
A= 0.99	0.23484	0.29341	0.39568	0.49162	0.56805	0.62679	0.65098	0.47442	0.54435
A= 1.00	0.24184	0.30231	0.40612	0.50162	0.57710	0.63491	0.65867	0.48368	0.55341

data by Chandrasekhar *et al.*, by Mayers, and by Bellman *et al.*, (see references) were good to at least four decimals. The functions $F_n(s, b)$, $G_{nm}(b)$ and $G'_{nm}(b)$, with n, m = 1 or 2, are implicitly contained in the lines marked "first order" (Table 8, page 44).

Finally, in Table 14 we present an excerpt of what happens to the flux (taken as 1000 units) when it strikes the atmosphere with different angular distributions. The fluxes emerging at top and bottom may be read from tabulations as just presented. The remainder, written on the middle lines is what is absorbed inside the atmosphere (Table 5, page 38).

TABLE 14: Table of Fluxes (Incident flux = 1000)

b = 0.5					b = 1.0			
GRAZ Z	THIN N	LAMB U	PERP $\mu_o = 1$		GRAZ Z	THIN N	LAMB U	PERP $\mu_o = 1$
				a = 0.2				
105	48	37	25	reflected	105	54	44	32
859	590	489	345	absorbed	878	771	710	576
36	362	474	630	transmitted	17	175	246	392
				a = 0.6				
353	176	137	93	reflected	363	209	174	130
509	364	303	214	absorbed	558	528	490	399
138	460	560	693	transmitted	79	263	336	471
				a = 0.95				
638	344	271	185	reflected	694	461	396	302
78	67	48	34	absorbed	97	99	93	75
284	599	681	781	transmitted	209	440	511	623
				a = 1				
687	375	296	203	reflected	758	516	447	341
0	0	0	0	absorbed	0	0	0	0
313	625	704	797	transmitted	242	484	553	659
				any a directly				
0	327	443	607	transmitted	0	148	219	368

ACKNOWLEDGEMENTS

This report was prepared during a stay at the Institute for Space Studies at New York made possible by a Senior Postdoctoral Fellowship from the National Aeronautics and Space Administration awarded by the National Academy of Sciences - National Research Council. The report contains some material of older date, but most of the developments and all numerical data are new. The work is being continued.

I gladly acknowledge the help derived from discussions with Drs. Arking, Lecar, Rasool, Spiegel and several other staff members and consultants at the Institute. Visits with Dr. Churchill, Dr. Greenberg, Dr. Hanel, Dr. Irvine, Dr. Kerker, and Dr. Nordberg have provided further stimulus.

I wish to extend my particular thanks to Dr. R. Jastrow, Director of the Institute for Space Studies, for his interest and hospitality, and to Mr. Grossman for invaluable help in performing the numerical computations.

REFERENCES

No attempt has yet been made to collect a complete bibliography. The following references have been used.

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